WELL-POSED EQUATIONS

M.Sc. MATHEMATICS

2024-2026

WELL-POSED EQUATIONS

DISSERTATION

Submitted to the University of Kerala in partial fulfillment of the requirements for the award of Master of Science in Mathematics

Programme Code	:	620
Exam Code	:	62023402
Course Code	:	MM 545

CERTIFICATE

This is to certify that the dissertation entitled **"WELL-POSED EQUATONS"** is a record of studies carried out by **NAYANA BABU**, M.Sc. Mathematics student of the year **2024–2026** at **T.K.M. College of Arts and Science, Kollam**, under my guidance and submitted to the University of Kerala in partial fulfillment of the Degree of Master of Science in Mathematics.

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Introduction

Numerical analysis is a branch of mathematics concerned with obtaining approximate solutions to mathematical problems that cannot be solved analytically. Many problems arising in science, engineering, and technology are formulated as operator equations, making approximation techniques essential for efficient computation. A fundamental concept in this area is that of a well-posed equation. According to Hadamard, an operator equation is said to be well-posed if it satisfies three important properties: the existence of a solution, the uniqueness of the solution, and the continuous dependence of the solution on the given data. These properties ensure that the problem is mathematically meaningful and that small changes in the input produce only small changes in the solution, leading to stable and reliable numerical computations.

When exact solutions are difficult or impossible to obtain, approximation methods are used to construct simpler problems whose solutions converge to the exact solution. The effectiveness of these methods depends on important concepts such as convergence, stability, norm approximation, and norm-square approximation, which ensure the accuracy and reliability of numerical algorithms. This project studies the theory of well-posed operator equations and their approximations by introducing the basic concepts of functional analysis and examining the convergence, stability, and approximation of bounded linear operators. These concepts provide a rigorous mathematical foundation for developing efficient numerical methods widely used in scientific computing and engineering applications

Chapter 1

Preliminaries

1.1 Basic Concepts and Definition

Definition 1.1.1 *A **linear space** (or* vector space*) over a field* $\mathbb{F}$ *(where* $\mathbb{F} = \mathbb{R}$ *or* $\mathbb{C}$*) is a non-empty set* X *together with two operations, namely vector addition and scalar multiplication, satisfying the following axioms for all* $x, y, z \in X$ *and* $\alpha, \beta \in \mathbb{F}$*:*

1. $x + y = y + x$
2. $(x + y) + z = x + (y + z)$
3. *There exists an element* $0 \in X$ *such that* $x + 0 = x$
4. *For every* $x \in X$*, there exists an element* $-x \in X$ *such that* $x + (-x) = 0$
5. $\alpha(x + y) = \alpha x + \alpha y$
6. $(\alpha + \beta)x = \alpha x + \beta x$
7. $\alpha(\beta x) = (\alpha\beta)x$

8. $1x = x$, *where* 1 *is the multiplicative identity in* $\mathbb{F}$.

Example 1.1.1 *The set* $\mathbb{R}^n$ *with the usual vector addition and scalar multiplication is a linear space over* $\mathbb{R}$.

Definition 1.1.2 *Let* X *be a vector space over* $\mathbb{R}$ *or* $\mathbb{C}$. *A function*

$$\|\cdot\| : X \to [0, \infty)$$

is called a ***norm*** *on* X *if, for every* $x, y \in X$ *and every scalar* α, *the following properties hold:*

1. ***Positive definiteness:***

$$\|x\| \geq 0, \quad \text{and } \|x\| = 0 \iff x = 0$$

2. ***Homogeneity:***

$$\|\alpha x\| = |\alpha| \, \|x\|$$

3. ***Triangle inequality:***

$$\|x + y\| \leq \|x\| + \|y\|$$

A vector space equipped with a norm is called a ***normed vector space*** *(or simply a* normed space*).*

Definition 1.1.3 *A normed linear space is a vector space* V *together with a norm* $\|\cdot\|$ *such that for all* $x, y \in V$ *and any scalar* α,

1. ***Non-negativity:***

$$\|x\| \geq 0, \qquad \|x\| = 0 \iff x = 0$$

2. ***Homogeneity:***

$$\|\alpha x\| = |\alpha|\,\|x\|$$

3. ***Triangle Inequality:***

$$\|x + y\| \leq \|x\| + \|y\|$$

If these three properties hold, then $(V, \|\cdot\|)$ is called a ***normed linear space****.*

Example 1.1.2 *Euclidean Space $\mathbb{R}^2$*

Let

$$V = \mathbb{R}^2 = \{(x, y) : x, y \in \mathbb{R}\}$$

Define the norm by

$$\|(x, y)\| = \sqrt{x^2 + y^2}$$

Definition 1.1.4 ***Banach space*** *is a complete normed linear space,i.e.,a normed linear space in which every Cauchy sequence converges to an element of the same space.*

Definition 1.1.5 *Let X and Y be linear spaces over the same field $\mathbb{F}$. A mapping*

$$T : X \to Y$$

is called a ***linear operator*** *if, for all $x, y \in X$ and $\alpha, \beta \in \mathbb{F}$,*

$$T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$$

Example 1.1.3 *Define $T : \mathbb{R}^2 \to \mathbb{R}^2$ by*

$$T(x, y) = (2x - y,\ x + 3y).$$

Then T is a linear operator.

Definition 1.1.6 *Let X and Y be normed linear spaces. A linear operator*

$$T : X \to Y$$

is called a ***bounded linear operator*** *if there exists a constant $M \geq 0$ such that*

$$\|Tx\| \leq M\|x\|, \qquad \forall\, x \in X$$

The set of all bounded linear operators from X into Y is denoted by

$$B(X, Y) = \{T : X \to Y : T \text{ is a bounded linear operator}\}$$

If $X = Y$, we write

$$B(X) = B(X, X)$$

Definition 1.1.7 *Let X and Y be normed linear spaces, and let $T : X \to Y$ be a bounded linear operator. The norm of T, called the* ***operator norm****, is defined by*

$$\|T\| = \sup\{\|Tx\| : \|x\| \leq 1\}$$

Equivalently,

$$\|T\| = \sup_{x \neq 0} \frac{\|Tx\|}{\|x\|}$$

Definition 1.1.8 *Let $T : X \to Y$ be a linear operator.The* ***range*** *(or* image*) of T is defined by*

$$R(T) = \{Tx : x \in X\}$$

That is,$R(T)$ is the set of all images of elements of X under the operator$T$. Let $T : X \to Y$ be a linear operator. The ***null space*** *(or* kernel*) of T is defined by*

$$N(T) = \{x \in X : Tx = 0\}$$

That is, $N(T)$ consists of all vectors in X that are mapped to the zero vector in Y.

Theorem 1.1.1 ***(Uniform Boundedness Principle)*** *Let X be a Banach space, Y be a normed space and $\mathcal{F}$ be a subset of $BL(X,Y)$ such that for each $x \in X$, the set*

$$\{F(x) : F \in \mathcal{F}\}$$

is bounded in Y. Then for each bounded subset E of X, the set

$$\{F(x) : x \in E,\, F \in \mathcal{F}\}$$

is bounded in Y, that is, $\mathcal{F}$ is uniformly bounded on E. In particular,

$$\sup\{\|F\| : F \in \mathcal{F}\} < \infty$$

Proof. *For $n = 1, 2, \ldots,$ let*

$$D_n = \{x \in X : \|F(x)\| > n \text{ for some } F \in \mathcal{F}\}$$

For each $F \in \mathcal{F}$, the function $x \mapsto \|F(x)\|$ is continuous on X, so that the set

$$\{x \in X : \|F(x)\| > n\}$$

is open in X. Since D_n is the union of all these sets, it follows that D_n is open in X. Let $x \in X$. Then, by hypothesis, $\|F(x)\| \leq n$ for all $F \in \mathcal{F}$ and some positive integer n, that is, $x \notin D_n$. Thus

$$\bigcap_{n=1}^{\infty} D_n = \varnothing$$

Consequently,

$$\bigcap_{n=1}^{\infty} D_n^c = X$$

Since X is a Banach space, Baire's theorem implies that some D_m^c must not be nowhere dense in X. Then there exist $a \in X$ and $r > 0$ such that

$$\overline{U}(a,r) \subset D_m^c,$$

that is, if $y \in X$ *and*

$$\|y - a\| \leq r,$$

then

$$\|F(y)\| \leq m \qquad \text{for all } F \in \mathcal{F}.$$

Theorem 1.1.2 ***(Squeeze Theorem)*** *Let* $a_n, b_n,$ *and* c_n *be three sequences of real numbers such that there exists* $N \in \mathbb{N}$ *for which*

$$a_n \leq b_n \leq c_n \quad \forall\, n \geq N,$$

and if

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L,$$

then

$$\lim_{n\to\infty} b_n = L$$

Definition 1.1.9 *Let* X *be a Banach space and let* X_n *be a subspace of* X*. A bounded linear operator*

$$P_n \in B(X)$$

is called a ***projection*** *onto* X_n *if*

$$R(P_n) = X_n$$

and

$$P_n x = x, \qquad \forall\, x \in X_n$$

Equivalently,

$$P_n^2 = P_n,$$

Definition 1.1.10 *Let* X *and* Y *be normed linear spaces and let*

$$T : X \to Y$$

be a linear operator. The operator T is said to be a ***compact operator*** *if for every bounded subset E of X, the set $T(E)$ has compact closure in Y. Equivalently, T is compact if for every bounded sequence $\{x_n\}$ in X, the sequence $\{Tx_n\}$ has a convergent subsequence in Y.*

Example 1.1.4 *Let $X = \ell^2$, and define the operator $T : \ell^2 \to \ell^2$ by*

$$T(x_1, x_2, x_3, \ldots) = \left(x_1, \frac{x_2}{2}, \frac{x_3}{3}, \ldots, \frac{x_n}{n}, \ldots\right)$$

Since $\frac{1}{n} \to 0$ as $n \to \infty$, the sequence of coefficients tends to zero. Hence, the image of every bounded sequence under T has a convergent subsequence. Therefore, T is a compact operator.

Theorem 1.1.3 ***(Neumann Series Theorem)*** *Let X be a Banach space and let $T \in B(X)$ be a bounded linear operator. If*

$$\|T\| < 1,$$

then the series

$$\sum_{n=0}^{\infty} T^n = I + T + T^2 + \cdots$$

converges in the operator norm to a bounded linear operator S. Moreover,

$$(I - T)^{-1} = \sum_{n=0}^{\infty} T^n$$

Hence, $I - T$ is invertible

Proof. Since $\|T\| < 1$, we have

$$\|T^n\| \le \|T\|^n, \qquad n = 0, 1, 2, \ldots$$

Hence,

$$\sum_{n=0}^{\infty} \|T^n\| \le \sum_{n=0}^{\infty} \|T\|^n = \frac{1}{1 - \|T\|} < \infty$$

Therefore, the series

$$\sum_{n=0}^{\infty} T^n$$

converges absolutely in the Banach space $B(X)$ of bounded linear operators.

Let

$$S = \sum_{n=0}^{\infty} T^n$$

Consider the partial sums

$$S_n = I + T + T^2 + \cdots + T^n$$

Then

$$(I - T)S_n = (I - T)(I + T + \cdots + T^n) = I - T^{n+1}$$

Since

$$\|T^{n+1}\| \leq \|T\|^{n+1} \longrightarrow 0 \quad \text{as } n \to \infty,$$

taking limits gives

$$(I - T)S = I$$

Similarly,

$$S_n(I - T) = I - T^{n+1},$$

and letting $n \to \infty$, we obtain

$$S(I - T) = I$$

Thus,

$$(I - T)S = S(I - T) = I,$$

which shows that S is the inverse of $I - T$. Hence,

$$(I - T)^{-1} = S = \sum_{n=0}^{\infty} T^n$$

Therefore, $I - T$ is invertible with inverse given by the Neumann series.

Chapter 2

Well-Posed Equations

2.1 Well-posed equations

Definition 2.1.1 *The operator equation*

$$Tx = y$$

is said to be ***well-posed*** *if*

$$T : X \to Y$$

is a bounded linear operator between the normed linear spaces X and Y such that T is bijective, and

$$T^{-1} : Y \to X$$

is continuous.

Example 2.1.1 *Consider the Fredholm integral equation of the second kind*

$$x(s) - \frac{1}{2}\int_0^1 x(t)\,dt = 1, \qquad s \in [0,1]$$

Define the operator

$$K : C[0,1] \to C[0,1]$$

by

$$(Kx)(s) = \frac{1}{2}\int_0^1 x(t)\,dt$$

Then the equation can be written as

$$(I - K)x = f,$$

where

$$f(s) = 1$$

Take any $x \in C[0,1]$. *Then*

$$(Kx)(s) = \frac{1}{2}\int_0^1 x(t)\,dt$$

Taking absolute values,

$$|(Kx)(s)| = \left|\frac{1}{2}\int_0^1 x(t)\,dt\right|$$

Since

$$|x(t)| \le \|x\|_\infty,$$

for every t,

$$\frac{1}{2}\int_0^1 |x(t)|\,dt \le \frac{1}{2}\|x\|_\infty$$

Hence,

$$|(Kx)(s)| \le \frac{1}{2}\|x\|_\infty$$

Therefore,

$$\|K\| \le \frac{1}{2} < 1$$

Since $\|K\| < 1$, the Neumann Series Theorem implies that $I - K$ is invertible and

$$(I - K)^{-1} = I + K + K^2 + K^3 + \cdots = \sum_{n=0}^{\infty} K^n$$

Moreover,

$$\|(I - K)^{-1}\| \leq \frac{1}{1 - \|K\|}$$

Since $I - K$ has an inverse, the equation

$$(I - K)x = f$$

has the unique solution

$$x = (I - K)^{-1} f$$

Also,

$$\|x\| = \|(I - K)^{-1} f\|$$

Since $(I - K)^{-1}$ is a bounded linear operator,

$$\|(I - K)^{-1} f\| \leq \|(I - K)^{-1}\| \, \|f\|$$

Hence,

$$\|x\| \leq \|(I - K)^{-1}\| \, \|f\|$$

Therefore, small changes in the data f produce only small changes in the solution x. Hence, the solution depends continuously on the given data.

2.2 Convergence

Consider an approximation method

$$\mathcal{A} = \{(X_n, Y_n, T_n, y_n) : n \in \mathbb{N}\}$$

where $X_n \subset X$ and $Y_n \subset Y$ and $T_n : X_n \to Y_n$ is a bounded linear operator and $y_n \in Y_n$ for the operator equation

$$Tx = y$$

is said to be **convergent** if, for all sufficiently large n, the approximate equation

$$T_n x_n = y_n$$

has a unique solution x_n, and

$$x_n \longrightarrow x \quad \text{as } n \to \infty,$$

where x is the unique solution of the operator equation $Tx = y$.

Definition 2.2.1 ***Stable***

A sequence (T_n) in $\mathcal{B}(X, Y)$ is said to be ***stable*** *with a stability index or simply an index $N \in \mathbb{N}$, if*

(a) $(\|T_n\|)$ is bounded,

(b) T_n is bijective for all $n \geq N$, and

(c) $\{\|T_n^{-1}\| : n \geq N\}$ is bounded.

Proposition 1 *Suppose there exists $N \in \mathbb{N}$ such that T_n is invertible in $\mathcal{B}(X, Y)$ for all $n \geq N$ and*

$$\|(T - T_n)x\| \to 0 \quad \textit{as } n \to \infty$$

for every $x \in X$. Then the following holds:

(i)If $\{\|T_n^{-1}\|\}_{n \geq N}$ is bounded, then T is injective.

(ii)If $y \in Y$ is such that $(T_n^{-1}y)$ converges, then $y \in R(T)$.

In particular, if (T_n) is stable,

$$\|(T - T_n)x\| \to 0 \quad \text{as } n \to \infty$$

for every $x \in X$, and $(T_n^{-1}y)$ converges for every $y \in Y$, then

$$Tx = y$$

is well-posed.

Proof: (i) Assume $Tx = 0$

Take some $x \in X$ such that $Tx = 0$

Since T_n is invertible,

$$x = T_n^{-1}(T_n x)$$

Taking norms,

$$\|x\| = \|T_n^{-1} T_n x\|$$

And since $Tx = 0$,

$$T_n x = (T_n - T)x$$

So,

$$\|x\| = \left\|T_n^{-1}(T_n - T)x\right\| \leq \|T_n^{-1}\| \, \|(T_n - T)x\|$$

Assume $\|T_n^{-1}\| \leq C$ for all $n \geq N$. Then

$$\|x\| \leq C \, \|(T_n - T)x\|$$

But

$$\|(T_n - T)x\| \to 0 \qquad \text{as } n \to \infty$$

So,

$$\|x\| \to 0$$

Hence $x = 0$

(ii) Since

$$\|(T - T_n)x\| \to 0 \quad \forall x$$

$$\Rightarrow \quad T_n x \to Tx \quad \forall x$$

Then by the Uniform Boundedness Principle, there exist $b > 0$ such that

$$\|T_n^{-1}\| \leq b \quad \forall n$$

Let $y \in Y$ be such that $(T_n^{-1}y)$ converges. Assume $T_n^{-1}y \to x \quad \forall x$
Since $y = T_n(T_n^{-1}y)$,

$$y - T_n x = T_n(T_n^{-1}y) - T_n x$$

$$y - T_n(x) = T_n(T_n^{-1}y - x)$$

Taking norms,

$$\|y - T_n(x)\| = \|T_n(T_n^{-1}y - x)\|$$

$$\leq \|T_n\| \, \|T_n^{-1}y - x\|$$

Since $\|T_n\| \leq b$ and $T_n^{-1}y \to x$, we get

$$\|y - T_n(x)\| \to 0$$

$$T_n(x) \to y$$

But we have $T_n x \to Tx$ Therefore,

$$y = Tx$$

Hence $y \in R(T)$

Theorem 2.2.1 *Suppose (T_n) is stable with the index $N \in \mathbb{N}$. Let x_n be the unique solution of*

$$T_n x_n = y_n$$

for $n \geq N$. Then for any $x \in X$,

$$c_1 \|T_n x - y_n\| \leq \|x - x_n\| \leq c_2 \|T_n x - y_n\|$$

where c_1 and c_2 are positive real numbers such that

$$\|T_n\| \leq \frac{1}{c_1} \quad \text{and} \quad \|T_n^{-1}\| \leq c_2, \qquad \forall\, n \geq N.$$

If, in addition, $x \in X$ and $y \in Y$ are such that

$$\alpha_n := \|(T - T_n)x + (y_n - y)\| \longrightarrow 0 \qquad \text{as } n \to \infty,$$

then

$$x_n \to x \iff Tx = y,$$

and in that case,

$$\alpha_n = \|T_n x - y_n\|$$

Proof. Let $n \geq N$. Since x_n is the unique solution of the equation

$$T_n x_n = y_n,$$

we have

$$T_n(x - x_n) = T_n x - y_n$$

Since T_n is invertible,

$$x - x_n = T_n^{-1}(T_n x - y_n)$$

Taking norms,

$$\|x - x_n\| = \|T_n^{-1}(T_n x - y_n)\| \leq \|T_n^{-1}\|\, \|T_n x - y_n\|$$

Since $\|T_n^{-1}\| \leq c_2$,

$$\|x - x_n\| \leq c_2\|T_n x - y_n\|$$

From equation

$$T_n x - y_n = T_n(x - x_n)$$

Hence,

$$\|T_n x - y_n\| = \|T_n(x - x_n)\| \leq \|T_n\| \, \|x - x_n\|$$

Since

$$\|T_n\| \leq \frac{1}{c_1},$$

we get,

$$\|T_n x - y_n\| \leq \frac{1}{c_1}\|x - x_n\|,$$

or equivalently,

$$c_1\|T_n x - y_n\| \leq \|x - x_n\|$$

Therefore,

$$c_1\|T_n x - y_n\| \leq \|x - x_n\| \leq c_2\|T_n x - y_n\|$$

This implies that

$$x_n \to x \iff \|T_n x - y_n\| \to 0$$

Define

$$\alpha_n = \|(T - T_n)x + (y_n - y)\|$$

We can see that,

$$(T - T_n)x + (y_n - y) = (Tx - y) - (T_n x - y_n)$$

Hence,

$$\alpha_n \to 0 \iff T_n x - y_n \to Tx - y$$

Since $\alpha_n \to 0$, it follows that

$$T_n x - y_n \to 0 \iff Tx - y = 0$$

Therefore,

$$x_n \to x \Longleftrightarrow Tx = y$$

If $Tx = y$, then

$$\alpha_n = \|(T - T_n)x + (y_n - y)\| = \|(Tx - y) - (T_n x - y_n)\| = \|T_n x - y_n\|$$

Corollary 2.2.1 *Suppose T is bijective and (T_n) is stable with index $N \in \mathbb{N}$. Let x be the solution of*

$$Tx = y,$$

and x_n be the solution of

$$T_n x_n = y_n, \qquad n \geq N.$$

Suppose further that

$$y_n \to y \quad \textit{and} \quad (T - T_n)x \to 0 \qquad \textit{as } n \to \infty$$

Then

$$\alpha_n := \|T_n x - y_n\| \to 0 \qquad \textit{as } n \to \infty,$$

and

$$c_1 \alpha_n \leq \|x - x_n\| \leq c_2 \alpha_n, \qquad \forall n \geq N,$$

where c_1 and c_2 are positive real numbers such that

$$\|T_n\| \leq \frac{1}{c_1} \textit{and} \qquad \|T_n^{-1}\| \leq c_2, \qquad \forall n \geq N$$

In particular,

$$x_n \to x \qquad \textit{as } n \to \infty$$

Proof. Since

$$Tx = y, \qquad T_n x_n = y_n,$$

we have

$$T_n(x - x_n) = T_n x - y_n$$

Hence,

$$x - x_n = T_n^{-1}(T_n x - y_n)$$

Taking norms on both sides,

$$\|x - x_n\| \leq \|T_n^{-1}\| \, \|T_n x - y_n\| = \|T_n^{-1}\|\alpha_n$$

Since (T_n) is stable,

$$\|T_n^{-1}\| \leq c_2,$$

we obtain

$$\|x - x_n\| \leq c_2\alpha_n$$

For the lower estimate, we can see that

$$T_n x - y_n = T_n(x - x_n)$$

Therefore,

$$\alpha_n = \|T_n(x - x_n)\| \leq \|T_n\| \, \|x - x_n\|$$

Since

$$\|T_n\| \leq \frac{1}{c_1},$$

it follows that

$$\alpha_n \leq \frac{1}{c_1}\|x - x_n\|,$$

$$c_1\alpha_n \leq \|x - x_n\|$$

We have

$$\alpha_n = \|T_n x - y_n\| = \|T_n x - Tx + y - y_n\|$$

Using the triangle inequality,

$$\alpha_n \leq \|(T_n - T)x\| + \|y_n - y\|$$

Since

$$(T - T_n)x \to 0 \quad \text{and} \quad y_n \to y,$$

implies that

$$\alpha_n \to 0$$

Combining the inequalities,

$$c_1\alpha_n \leq \|x - x_n\| \leq c_2\alpha_n,$$

and using the fact that $\alpha_n \to 0$, the Squeeze Theorem implies

$$\|x - x_n\| \to 0$$

Hence,

$$x_n \to x$$

Theorem 2.2.2 *Assume that (T_n) is stable with index N and let x_n be the solution of*

$$T_n x_n = y_n$$

for all $n \geq N$. Then for any $x \in X$,

$$\|x - x_n\| \leq \|x - P_n x\| + c\|T_n P_n x - y_n\|,$$

where $c \geq \|T_n^{-1}\|$ for all $n \geq N$. Further, if $x \in X$ and $y \in Y$ are such that

$$\tilde{\alpha}_n := \|(T_n P_n - T)x + (y - y_n)\| \to 0$$

and $P_n x \to x$ as $n \to \infty$, then

$$x_n \to x \iff Tx = y,$$

and in that case, $\tilde{\alpha}_n = \|T_n P_n x - y_n\|$

Proof. Since x_n is the solution of

$$T_n x_n = y_n,$$

and $P_n : X \to X_n$ is a projection. For $x \in X$, we have

$$x - x_n = (x - P_n x) + (P_n x - x_n)$$

Since T_n is invertible,

$$P_n x - x_n = T_n^{-1}(T_n P_n x - T_n x_n) = T_n^{-1}(T_n P_n x - y_n)$$

Therefore,

$$x - x_n = (x - P_n x) + T_n^{-1}(T_n P_n x - y_n) \tag{1}$$

Taking norms,

$$\begin{aligned} \|x - x_n\| &= \left\|(x - P_n x) + T_n^{-1}(T_n P_n x - y_n)\right\| \\ &\leq \|x - P_n x\| + \left\|T_n^{-1}(T_n P_n x - y_n)\right\| \\ &\leq \|x - P_n x\| + \|T_n^{-1}\| \, \|T_n P_n x - y_n\| \end{aligned}$$

Since (T_n) is stable,

$$c \geq \|T_n^{-1}\|,$$

and hence

$$\|x - x_n\| \leq \|x - P_n x\| + c \, \|T_n P_n x - y_n\|$$

Now suppose

$$\tilde{\alpha}_n = \|(T_n P_n - T)x + (y - y_n)\| \longrightarrow 0,$$

and

$$P_n x \longrightarrow x$$

We know that,

$$T_n P_n x - y_n = (T_n P_n - T)x + (y - y_n) + (Tx - y)$$

Hence,

$$T_nP_nx - y_n \longrightarrow Tx - y$$

Using (1),

$$x - x_n = (x - P_nx) + T_n^{-1}(T_nP_nx - y_n)$$

Since (T_n) is stable, the sequence $\{T_n^{-1}\}$ is uniformly bounded. If

$$Tx = y,$$

then

$$T_nP_nx - y_n \longrightarrow 0$$

Also,

$$P_nx \longrightarrow x$$

Hence,

$$x_n \longrightarrow x$$

Conversely, if

$$x_n \longrightarrow x$$

Then from (1),

$$P_nx - x_n \longrightarrow 0,$$

because

$$P_nx \longrightarrow x$$

Therefore,

$$T_nP_nx - y_n = T_n(P_nx - x_n) \longrightarrow 0,$$

since the operators T_n are uniformly bounded. But we already know that

$$T_nP_nx - y_n \longrightarrow Tx - y$$

Hence,

$$Tx - y = 0,$$

that is,

$$Tx = y$$

Thus,

$$x_n \longrightarrow x \iff Tx = y$$

When $Tx = y$,

$$(T_nP_n - T)x + (y - y_n) = T_nP_nx - y_n,$$

so that

$$\tilde{\alpha}_n = \|T_nP_nx - y_n\|$$

Corollary 2.2.2 *Suppose T is bijective and (T_n) is stable with index N. Let x be the solution of*

$$Tx = y$$

and x_n be the solution of

$$T_nx_n = y_n$$

for all $n \geq N$. Suppose further that

$$y_n \to y, \qquad P_nx \to x, \qquad (T_nP_n - T)x \to 0$$

as $n \to \infty$. Then

$$\|x - x_n\| \leq \|x - P_nx\| + c\|T_nP_nx - y_n\|,$$

where $c > 0$ is such that $\|T_n^{-1}\| \leq c$ for all $n \geq N$. In particular,

$$x_n \to x \qquad as \qquad n \to \infty.$$

2.3 Conditions for Stability

In this section we discuss some sufficient conditions on the approximation properties of the sequence (T_n) of operators which ensure its stability when $X_n = X$ and $Y_n = Y$. A simplest assumption that one may impose on (T_n) would be to have *pointwise convergence*, that is, for each $x \in X$,

$$\|Tx - T_n x\| \longrightarrow 0 \qquad \text{as } n \to \infty.$$

In that case, we also say that (T_n) is a *pointwise approximation* of T, and write

$$T_n \longrightarrow T \qquad \text{pointwise.}$$

Moreover, if (T_n) is a pointwise approximation of T, then by the Banach–Steinhaus theorem, the sequence of operator norms $(\|T_n\|)$ is bounded.

Example 2.3.1 *Suppose $T \in \mathcal{B}(X,Y)$, and (P_n) and (Q_n) are projection operators in $\mathcal{B}(X)$ and $\mathcal{B}(Y)$, such that for each $x \in X$ and $y \in R(T)$,*

$$P_n x \to x \quad \text{and} \quad Q_n y \to y \qquad \text{as } n \to \infty.$$

Then we have

$$\|TP_n x - Tx\| = \|T(P_n x - x)\| \leq \|T\|\,\|P_n x - x\| \longrightarrow 0,$$

and

$$\|Q_n Tx - Tx\| \longrightarrow 0$$

as $n \to \infty$ for every $x \in X$. Thus, (TP_n) and $(Q_n T)$ are pointwise approximations of T. Suppose, in addition, that $(\|Q_n\|)$ is bounded. Then we have

$$\|Q_n TP_n x - Tx\| \leq \|Q_n(TP_n x - Tx)\| + \|Q_n Tx - Tx\|$$

$$\leq \|Q_n\| \, \|TP_nx - Tx\| + \|Q_nTx - Tx\| \longrightarrow 0$$

as $n \to \infty$ *for every* $x \in X$*. Thus, in this case,* (Q_nTP_n) *is also a pointwise approximation of* T*.*

Chapter 3

Norm Approximation

3.1 Norm approximation

Definition 3.1.1 *A sequence (T_n) in $\mathcal{B}(X, Y)$ is called a* ***norm approximation*** *of T if*

$$\|T - T_n\| \longrightarrow 0 \qquad \text{as } n \to \infty.$$

If (T_n) is a norm approximation of T, then we may also say that (T_n) converges to T in norm, and write

$$T_n \longrightarrow T$$

in norm.

Remark 3.1.1 *We may observe that if*

$$T : X \to X$$

is of the form

$$T = \lambda I - A$$

for some nonzero scalar λ, *and if* (A_n) *in* $\mathcal{B}(X)$ *is a norm approximation of* $A \in \mathcal{B}(X)$, *then taking*

$$T_n := \lambda I - A_n,$$

we have

$$\|T - T_n\| = \|A - A_n\| \longrightarrow 0 \qquad \text{as } n \to \infty.$$

Remark 3.1.2 *Let* X *be a normed linear space,* Y *be a Banach space and* $T : X \to Y$ *be a bijective linear operator with* $T^{-1} \in \mathcal{B}(Y, X)$. *If* $E : X \to Y$ *is a linear operator such that* $ET^{-1} \in \mathcal{B}(Y)$ *and* $\|ET^{-1}\| < 1$, *then* $T + E$ *is bijective,* $(T + E)^{-1} \in \mathcal{B}(Y, X)$ *and*

$$\|(T + E)^{-1}\| \leq \frac{\|T^{-1}\|}{1 - \|ET^{-1}\|}$$

Theorem 3.1.1 *Suppose* T *is bijective and* (T_n) *is a norm approximation of* T. *Then* (T_n) *is stable.*

Proof. For $n \in \mathbb{N}$, let

$$T_n = T + E_n,$$

where

$$E_n = T_n - T$$

Since $T_n \to T$ in norm,

$$\|T_n - T\| \longrightarrow 0,$$

that is,

$$\|E_n\| \longrightarrow 0.$$

Hence,

$$\|E_n T^{-1}\| \longrightarrow 0,$$

because T^{-1} is bounded. Since

$$\|E_n T^{-1}\| \longrightarrow 0,$$

there exists $N \in \mathbb{N}$ such that

$$\|E_n T^{-1}\| \leq \frac{1}{2},$$

for all $n \geq N$ Let X be a normed linear space, Y be a Banach space, and $T : X \to Y$ be a bijective linear operator with $T^{-1} \in \mathcal{B}(Y, X)$. If $E : X \to Y$ is a linear operator such that $ET^{-1} \in \mathcal{B}(Y)$ and

$$\|ET^{-1}\| < 1,$$

then $T + E$ is bijective,

$$(T + E)^{-1} \in \mathcal{B}(Y, X),$$

and

$$\|(T + E)^{-1}\| \leq \frac{\|T^{-1}\|}{1 - \|ET^{-1}\|}$$

$T_n = T + E_n$ is invertible. Hence, for all $n \geq N$, T_n is bijective and

$$\|T_n^{-1}\| \leq \frac{\|T^{-1}\|}{1 - \|E_n T^{-1}\|}$$

Now, since

$$\|E_n T^{-1}\| \leq \frac{1}{2},$$

we have

$$1 - \|E_n T^{-1}\| \geq \frac{1}{2}$$

Therefore,

$$\|T_n^{-1}\| \leq \frac{\|T^{-1}\|}{1/2} = 2\|T^{-1}\|$$

Thus,

$$\left(\|T_n^{-1}\|\right)$$

is bounded.

Corollary 3.1.1 *Suppose T is bijective and (T_n) is a norm approximation of T. Then there exists $N \in \mathbb{N}$ such that*

$$T_n x_n = y_n$$

is uniquely solvable for all $n \geq N$, and

$$\|x - x_n\| \leq c\|(T - T_n)x\| + \|y - y_n\| \qquad \forall n \geq N,$$

where c is a positive number and x and x_n are the solutions of

$$Tx = y$$

and

$$T_n x_n = y_n$$

respectively. In particular, if $y_n \to y$ as $n \to \infty$, then

$$x_n \to x \quad as \quad n \to \infty.$$

Proof. Suppose T is bijective and $T_n \to T$ in norm.
i.e. $\|T - T_n\| \to 0$ Then there exists some N such that for all $n \geq N$,

$$T_n x_n = y_n$$

has a unique solution x_n. From Theorem 2.2.2, we have

$$\|x - x_n\| \leq \|x - P_n x\| + c\,\|T_n P_n x - y_n\|$$

In this corollary we are in the case $X_n = X$. So, choose $P_n = I$, then $P_n x = x$. Hence $\|x - P_n x\| = 0$. Substituting this into the estimate, we get

$$\|x - x_n\| \leq c\,\|T_n x - y_n\| \tag{1}$$

Rewrite

$$T_n x - y_n = (T_n x - Tx) + (Tx - y_n) = (T_n x - Tx) + (y - y_n) = -(T - T_n)x + (y - y_n)$$

Taking norms,

$$\|T_n x - y_n\| \leq \|(T - T_n)x\| + \|y - y_n\|$$

Substitute into (1):

$$\|x - x_n\| \leq c\Big(\|(T - T_n)x\| + \|y - y_n\|\Big)$$

Since $T_n \to T$ in norm,

$$\|(T - T_n)x\| \to 0,$$

and if $y_n \to y$,

$$\|y - y_n\| \to 0$$

This implies $x_n \to x$.

Remark 3.1.3 *Let X be a Banach space, Y be a normed linear space, and (T_n) be a sequence of operators in $B(X, Y)$ which converges pointwise on X. Let $T : X \to Y$ be defined by*

$$Tx = \lim_{n\to\infty} T_n x, \qquad x \in X,$$

and let $S \subseteq X$ be such that $\mathrm{cl}(S)$ is compact. Then

$$\sup_{x\in S} \|T_n x - Tx\| \longrightarrow 0 \qquad \text{as } n \to \infty.$$

In particular, if Z is a normed linear space and $K : Z \to X$ is a compact operator, then

$$\|(T_n - T)K\| \longrightarrow 0 \qquad \text{as } n \to \infty.$$

Example 3.1.1 *Suppose $P_n \in \mathcal{B}(X)$ and $Q_n \in \mathcal{B}(Y)$ are projection operators such that $P_n x \to x$ and $Q_n y \to y$ as $n \to \infty$ for each $x \in X$ and $y \in Y$. Let $K : X \to X$ be a compact operator. Then by Remark 3.1.3,*

$$\|Q_n K - K\| \to 0 \quad \text{as} \quad n \to \infty$$

Thus, (Q_nK) is a norm approximations of K. Now, suppose X and Y are Hilbert spaces, and (P_n) is a sequence of orthogonal projections. Then by compactness of K^*, the adjoint of K, we have

$$\|KP_n - K\| = \|P_nK^* - K^*\| \to 0$$

Hence, by the boundedness of $(\|Q_n\|)$, we also have

$$\|Q_nKP_n - K\| \leq \|Q_n(KP_n - K)\| + \|Q_nK - K\|$$

$$\leq \|Q_n\| \, \|KP_n - K\| + \|Q_nK - K\|$$

$$\to 0$$

as $n \to \infty$. Thus, in this case, (KP_n) and (Q_nKP_n) are also norm approximations of K. Next, suppose that $X = Y$,

$$T = \lambda I - K \quad \text{and} \quad T_n : \lambda I - K_n,$$

where λ is a nonzero scalar and K_n belongs to $\{Q_nK, \, KP_n, \, Q_nKP_n\}$. By Remark 3.1.1, it follows that (T_n) is a norm approximation of T if $K_n = Q_nK$ or if X is a Hilbert space, P_n are orthogonal projections and K_n belongs to $\{KP_n, \, Q_nKP_n\}$.

Remark 3.1.4 ***(a)*** *Recall that if P is a projection, then $I - P$ is also a projection. Hence, $\|I - P\| \geq 1$ whenever $P \neq I$. Thus, if (P_n) is a sequence of projections such that $P_n \neq I$ for infinitely many n, then (P_n) cannot be a norm approximation of I.*

(b) *Suppose $T \in \mathcal{B}(X, Y)$ is invertible, and (P_n) and (Q_n) are sequences of (non-identity) projections in $\mathcal{B}(X)$ and $\mathcal{B}(Y)$ respectively. Then neither (Q_nT) nor (TP_n) is a norm approximation of T.*

3.2 Norm-square approximation

Definition 3.2.1 *Suppose $T \in \mathcal{B}(X,Y)$ is bijective. Then a sequence (T_n) in $\mathcal{B}(X,Y)$ is called a* ***norm-square approximation*** *of T if*

$$\left\| \left[(T - T_n) T^{-1} \right]^2 \right\| \to 0 \quad as \quad n \to \infty.$$

If (T_n) is a norm-square approximation of T, we also say that (T_n) converges to T in norm-square sense.

Example 3.2.1 *Let $X = K^2$ with any norm, and let*

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \qquad C = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}$$

Then we have

$$A - B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} = C$$

Since $A = I$,

$$A^{-1} = I$$

Hence,

$$(A - B)A^{-1} = C$$

Now,

$$C^2 = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Therefore,

$$\left[(A-B)A^{-1}\right]^2 = C^2 = 0$$

Now, for $n \in \mathbb{N}$, let T, T_n, and E be the operators corresponding to the matrices A, B, and C, respectively. Then,

$$T = A = I, \qquad T_n = B$$

Hence,

$$T - T_n = E$$

Also,

$$\left[(T-T_n)T^{-1}\right]^2 = E^2 = 0$$

Thus,

$$\left\| \left[(T-T_n)T^{-1}\right]^2 \right\| = \|E^2\| = 0, \qquad \forall n \in \mathbb{N}$$

Therefore, (T_n) is a norm-square approximation of T, but it is not a norm approximation.

Example 3.2.2 *Let $X = Y = \ell^2$, $T = I$ and $T_n := I - A_n$, where A_n is defined by*

$$A_n x = x(2)e_1 + \frac{x(n)}{n}e_n, \qquad x \in \ell^2$$

Since $T = I$, $T_n = I - A_n$, we get

$$T - T_n = I - (I - A_n) = A_n$$

Taking norms,

$$\|T - T_n\| = \|A_n\|$$

Take $x = e_2$. *Then*

$$A_n e_2 = e_1$$

Hence,

$$\|A_n e_2\| = 1$$

Since $\|e_2\| = 1$,

$$\|A_n\| \geq 1$$

On the other hand,

$$A_n x = x(2)e_1 + \frac{x(n)}{n} e_n$$

Therefore,

$$\|A_n x\|^2 = |x(2)|^2 + \frac{|x(n)|^2}{n^2} \leq |x(2)|^2 + |x(n)|^2 \leq \|x\|^2$$

Thus, $\|A_n\| \leq 1$ *Combining both inequalities, we get*

$$\|A_n\| = 1$$

Hence, $\|T - T_n\| = 1$, $\forall n \in \mathbb{N}$ *Take any* $x \in \ell^2$. *We have*

$$A_n x = x(2)e_1 + \frac{x(n)}{n} e_n$$

Apply A_n *again. The second coordinate of* $A_n x$ *is zero and the* n*-th coordinate of* $A_n x$ *equals* $\dfrac{x(n)}{n}$. *Hence,*

$$A_n^2 x = \frac{1}{n}\left(\frac{x(n)}{n}\right) e_n = \frac{x(n)}{n^2} e_n$$

Therefore,

$$A_n^2 = \frac{1}{n^2} P_n,$$

where P_n *is the projection onto the* n*-th coordinate. Since* $\|P_n\| = 1$,

$$\|A_n^2\| = \frac{1}{n^2}$$

Since $T = I$, we have $T^{-1} = I$. Therefore

$$(I - T_n)T^{-1} = A_n$$

Hence,

$$\|(T - T_n)T^{-1}\|^2 = \|A_n^2\| = \frac{1}{n^2} \to 0$$

Remark 3.2.1 *Let X be a Banach space and $A \in \mathcal{B}(X)$. If $\|A\| < 1$, then $I - A$ is bijective and*

$$\|(I - A)^{-1}\| \leq \frac{1}{1 - \|A\|}$$

Theorem 3.2.1 *Suppose T is bijective, (T_n) is a norm-square approximation of T and $(\|T_n\|)$ is bounded. Then (T_n) is stable.*

Proof. For $n \in \mathbb{N}$, let $T_n = T + E_n$, where $E_n = T_n - T$. Since $(\|T_n\|)$ is bounded, there exists $c > 0$ such that $\|E_n T^{-1}\| \leq c$ for all $n \in \mathbb{N}$. Also, the hypothesis $\|(E_n T^{-1})^2\| \to 0$ implies that there exists a positive integer N such that

$$\|(E_n T^{-1})^2\| \leq 1/2 \quad \forall\, n \geq N$$

Therefore, by Remark 3.2.1, the operator T_n is bijective and

$$\|T_n^{-1}\| \leq \frac{\|T^{-1}\|\,(1 + \|E_n T^{-1}\|)}{1 - \|(E_n T^{-1})^2\|} \leq 2(1 + c)\|T^{-1}\|$$

for all $n \geq N$. Thus, (T_n) is stable.

3.3 Applications of well-posed problems

1. **Weather Forecasting**

   Weather forecasting is based on differential equations of the form

   $$\frac{\partial u}{\partial t} = F(u), \qquad u(x,0) = u_0(x),$$

   where $u(x,t)$ represents atmospheric variables such as temperature, pressure, wind speed, and humidity. A well-posed problem satisfies

   $$\|u(t) - v(t)\| \le C\|u_0 - v_0\|,$$

   which means that a small change in the initial weather data produces only a small change in the predicted weather. Hence, well-posed equations provide stable and reliable weather forecasts.

2. **Heat Conduction**

   Heat conduction is described by the heat equation

   $$\frac{\partial u}{\partial t} = k\nabla^2 u, \qquad u(x,0) = u_0(x),$$

   where $u(x,t)$ denotes the temperature distribution and k is the thermal conductivity. For a well-posed problem,

   $$\|u(t) - v(t)\| \le C\|u_0 - v_0\|,$$

   which ensures that small changes in the initial temperature cause only small changes in the temperature distribution. Therefore, well-posed equations guarantee stable and accurate prediction of heat transfer in engineering and physics.

Conclusion

This project has presented the fundamental concepts of well-posed operator equations and their approximations. A well-posed equation is characterized by the existence, uniqueness, and continuous dependence of the solution on the given data, ensuring that numerical computations are stable and reliable. The concepts of convergence and stability were studied to show how approximation methods produce solutions that approach the exact solution while controlling computational errors. The project also discussed norm and norm-square approximations, which provide effective techniques for approximating bounded linear operators and preserving stability. Overall, these concepts form a strong theoretical foundation for developing accurate and efficient numerical methods that are widely used in scientific computing, engineering, and applied mathematics.

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