ENUMERATION OF FUZZY SUBGROUPS IN FINITE CYCLIC AND SYMMETRIC GROUPS

M.Sc. MATHEMATICS

2024-2026

Scanned with OKEN Scanner

ENUMERATION OF FUZZY SUBGROUPS IN FINITE CYCLIC AND SYMMETRIC GROUPS

DISSERTATION

Submitted to the University of Kerala in partial fulfillment of the requirements for the award of Master of Science in Mathematics

Programme Code	:	620
Exam Code	:	62023402
Course Code	:	MM 545

2

Scanned with OKEN Scanner

CERTIFICATE

This is to certify that the dissertation entitled **"ENUMERATION OF FUZZY SUBGROUPS IN FINITE CYCLIC AND SYMMETRIC GROUPS"** is a record of studies carried out by **SHERIN N**, M.Sc. Mathematics student of the year **2024-2026** at **T.K.M. College of Arts and Science, Kollam**, under my guidance and submitted to the University Of Kerala in partial fulfillment of the Degree of Master of Science in Mathematics.

Kollam

July 2026

For

NIZA N

Assistant Professor

Department of Mathematics

T.K.M. College of Arts and Science

Dr. ASWATHY M R

H.O.D. and Assistant Professor

Department of Mathematics

T.K.M. College of Arts and Science

Scanned with OKEN Scanner

Contents

Scanned with OKEN Scanner

Introduction

Fuzzy set theory, introduced by Lotfi A. Zadeh in 1965, has an important role for situations involving uncertainty. Its ideas have been extended to various branches of mathematics, including algebra, leading to the development of fuzzy algebraic structures. In particular, the concept of a fuzzy subgroup allows each element of a group to have a degree of membership value between 0 and 1.

The study of fuzzy subgroups is related to the subgroup structure of the underlying group. In many cases, different fuzzy subgroups produce the same collection of level subgroups, making it useful to classify them through an equivalence relation. This classification establishes a one-to-one correspondence between the equivalence classes of fuzzy subgroups and the chains of ordinary subgroups. As a result, the problem of counting fuzzy subgroups can be reduced to counting suitable subgroup chains.

This project focuses on the study of fuzzy groups and the enumeration of fuzzy subgroups of finite cyclic and symmetric groups. After introducing the necessary definitions and preliminary results, the equivalence relation on fuzzy subgroups and its relation with subgroup chains are discussed. These ideas are then used to determine the number of distinct fuzzy subgroups of selected finite symmetric groups.

Scanned with OKEN Scanner

Chapter 1

Preliminaries

In 1965, Zadeh mathematically formulated the fuzzy subset concept. He defined a fuzzy subset of a non-empty set as a collection of objects with membership values, such that each object being assigned a value between 0 and 1 by a membership function. Fuzzy set theory was guided by the assumption that classical sets were not natural, appropriate or useful notions in describing the real-life problems, because every object encountered in this real physical world carries some degree of fuzziness.

1.1 Fuzzy Subsets

Definition 1.1.1. *A fuzzy subset μ of a non-empty set X is a function*

$$\mu : X \to [0,1].$$

Definition 1.1.2. *Let μ be a fuzzy subset of X. For $t \in [0,1]$, the set*

$$\mu_t = \{x \in X : \mu(x) \geq t\}$$

is called the t-level subset of μ.

Definition 1.1.3. *A fuzzy set μ of a set X is called a fuzzy point if and only if it takes the value 0 for all $y \in X$ except one, say $x \in X$. If its value at x is t, where $0 < t \leq 1$, then this fuzzy point is denoted by x_t.*

That is,

$$\mu(y) = \begin{cases} t, & \text{if } y = x, \quad 0 < t \leq 1, \\ 0, & \text{if } y \neq x. \end{cases}$$

Definition 1.1.4. *Let λ and μ be two fuzzy subsets of a set X. Then λ is said to be a* subset *of μ, written as*

$$\lambda \subseteq \mu,$$

if

$$\lambda(x) \leq \mu(x), \quad \forall x \in X.$$

If

$$\lambda(x) = \mu(x), \quad \forall x \in X,$$

then λ and μ are said to be equal, and we write

$$\lambda = \mu.$$

Scanned with OKEN Scanner

Definition 1.1.5. *Let λ and μ be fuzzy subsets of the sets X and Y, respectively. The Cartesian product of λ and μ is the fuzzy subset*

$$\lambda \times \mu : X \times Y \to [0,1]$$

defined by

$$(\lambda \times \mu)(x,y) = \min\{\lambda(x),\, \mu(y)\}$$

for every $(x,y) \in X \times Y$.

Definition 1.1.6. *Let λ and μ be two fuzzy subsets of a set X. The union of λ and μ, denoted by $\lambda \cup \mu$, is the fuzzy subset of X defined by*

$$(\lambda \cup \mu)(x) = \max\{\lambda(x),\, \mu(x)\},$$

for every $x \in X$.

Definition 1.1.7. *Let μ and λ be fuzzy subsets of a set X. The* intersection *of λ and μ, denoted by $\lambda \cap \mu$, is a fuzzy subset of X defined as,*

$$(\lambda \cap \mu)(x) = \min\{\lambda(x),\, \mu(x)\}, \qquad \forall x \in X.$$

Definition 1.1.8. *A fuzzy subset μ of a set X is said to be* normal *if*

$$\sup\{\mu(x) : x \in X\} = 1.$$

Definition 1.1.9. *Two fuzzy subsets μ and λ of a set X are said to be* disjoint *if there exists no $x \in X$ such that $\mu(x) = \lambda(x)$*

Definition 1.1.10. *Let μ be a fuzzy subset of a set X. The complement of μ, denoted by μ^c, is defined by*

$$\mu^c(x) = 1 - \mu(x),$$

for every $x \in X$.

Definition 1.1.11. *Let X be any set. A fuzzy subset μ of X is said to have the* supremum property *if, for every subset A of X, there exists an element $x_0 \in A$ such that*

$$\mu(x_0) = \sup\{\mu(x) : x \in A\}.$$

Definition 1.1.12. *Let $f : X \to Y$ be a mapping.*

If μ is a fuzzy subset of Y, then the inverse image of μ under f, denoted by $f^{-1}(\mu)$, is defined by

$$(f^{-1}(\mu))(x) = \mu(f(x)), \qquad \forall\, x \in X.$$

Similarly, if λ is a fuzzy subset of X, then the image of λ under f is the fuzzy subset of Y given by

$$(f(\lambda))(y) = \begin{cases} \sup \lambda(x), & \textit{if } f(z) = y, z \in X \\ 0, & \textit{otherwise} \end{cases}$$

for every $y \in Y$.

Scanned with OKEN Scanner

Definition 1.1.13. *Let R_λ be a fuzzy binary relation on a set X. A fuzzy subset μ of X is said to be a pre-class of R_λ if*

$$\min\{\mu(x), \mu(y)\} \leq R_\lambda(x, y),$$

for every $x, y \in X$.

A fuzzy binary relation R_λ on X is said to be a similarity relation if it satisfies the following conditions for all $x, y, z \in X$:

1. $R_\lambda(x, x) = 1,$
2. $R_\lambda(x, y) = R_\lambda(y, x),$
3. $\min\{R_\lambda(x, y),\ R_\lambda(y, z)\} \leq R_\lambda(x, z).$

If $\mu(x) = 0$ for every $x \in X$, then μ is called the empty fuzzy set and is denoted by ϕ_X.

If $\mu(x) = 1$ for every $x \in X$, then μ is called the whole fuzzy set and is denoted by 1_X.

Scanned with OKEN Scanner

Chapter 2

Groups and Fuzzy Subgroups

The concept of fuzzy groups was introduced by Rosenfeld, who extended the theory of fuzzy sets to group theory. He showed that many classical results of group theory can be generalized with fuzzy sets. The fundamental idea behind fuzzy groups is to associate each element of a group with a membership value in the interval $[0, 1]$, thereby providing a fuzzy algebraic structure in which the level subsets of a fuzzy subgroup form ordinary subgroups of the underlying group.

In this section, we introduce the basic definitions and fundamental results concerning fuzzy groups and fuzzy subgroups that will be used throughout this work.

2.1 Fuzzy Subgroups

Definition 2.1.1. *Let G be a group. A fuzzy subset μ of G is called a* fuzzy subgroup *of G if*

1. $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$, *for all $x, y \in G$, and*
2. $\mu(x^{-1}) = \mu(x)$, *for all $x \in G$.*

Definition 2.1.2. *Let G be a group and let μ be a fuzzy subgroup of G. Then μ is called a* normal fuzzy subgroup *of G if*

$$\mu(y^{-1}xy) = \mu(x),$$

for all $x, y \in G$.

Definition 2.1.3. *Let A be a fuzzy subset of a group G. For any $t \in [0, 1]$, the set*

$$A_t = \{\, x \in G \mid A(x) \geq t \,\}$$

*is called the t-*level subset *(or t-*cut*) of the fuzzy subset A.*

Theorem 2.1.1. *Let G be a group and let A be a fuzzy subgroup of G. Then, for every $t \in [0, 1]$,$t \leq A(e)$ where e is the identity element of G, the level subset*

$$A_t = \{x \in G \mid A(x) \geq t\}$$

is a subgroup of G.

Scanned with OKEN Scanner

Proof. Let $t \in [0,1]$,$t \leq A(e)$ We show that A_t satisfies the subgroup conditions, Since $A(e) \geq t$, we have $e \in A_t$. Hence, A_t is non-empty.

Let $x, y \in A_t$. Then

$$A(x) \geq t \quad \text{and} \quad A(y) \geq t.$$

Since A is a fuzzy subgroup,

$$A(xy) \geq \min\{A(x), A(y)\} \geq t.$$

Thus, $xy \in A_t$. Hence, A_t is closed.

Now let $x \in A_t$. Since

$$A(x^{-1}) = A(x),$$

we have

$$A(x^{-1}) \geq t.$$

Therefore, $x^{-1} \in A_t$. Thus, A_t is closed under inverses.

Hence, A_t is non-empty and is closed under the group operation and inverses. Therefore, by the subgroup conditions, A_t is a subgroup of G. □

Theorem 2.1.2. *A fuzzy subset μ of a group G is a fuzzy subgroup of G if and only if*

$$\mu(xy^{-1}) \geq \min\{\mu(x), \mu(y)\}$$

for every $x, y \in G$.

Proof. Suppose μ is a fuzzy subgroup of G. Then, by definition,

$$\mu(ab) \geq \min\{\mu(a), \mu(b)\}$$

for all $a, b \in G$, and

$$\mu(x^{-1}) = \mu(x)$$

for every $x \in G$.

Let $a = x$ and $b = y^{-1}$. Then

$$\mu(xy^{-1}) \geq \min\{\mu(x), \mu(y^{-1})\}.$$

Since $\mu(y^{-1}) = \mu(y)$, we obtain

$$\mu(xy^{-1}) \geq \min\{\mu(x), \mu(y)\}.$$

Conversely, suppose that

$$\mu(xy^{-1}) \geq \min\{\mu(x), \mu(y)\}$$

for every $x, y \in G$.

Putting $x = e$ and $y = e$, we get

$$\mu(e) \geq \mu(e),$$

which is trivially true.

Now, let $x = e$. Then

$$\mu(y^{-1}) \geq \min\{\mu(e), \mu(y)\} = \mu(y),$$

Scanned with OKEN Scanner

since $\mu(e) \geq \mu(y)$ for every $y \in G$.

Replacing y by y^{-1}, we obtain

$$\mu(y) \geq \mu(y^{-1}).$$

hence,

$$\mu(y) = \mu(y^{-1}).$$

Now replace y by y^{-1} , we have

$$\mu(xy) \geq \min\{\mu(x), \mu(y^{-1})\} = \min\{\mu(x), \mu(y)\}.$$

Therefore,

$$\mu(xy) \geq \min\{\mu(x), \mu(y)\}$$

and

$$\mu(y^{-1}) = \mu(y).$$

hence, μ is a fuzzy subgroup of G. □

Theorem 2.1.3. *Let μ be a fuzzy subgroup of a group G. Then*

$$\mu(e) \geq \mu(x),$$

for every $x \in G$, where e is the identity element of G.

Proof. Let $x \in G$. Since μ is a fuzzy subgroup of G,

$$\mu(xy^{-1}) \geq \min\{\mu(x), \mu(y)\},$$

for all $x, y \in G$.

Putting $y = x$, we obtain

$$\mu(xx^{-1}) \geq \min\{\mu(x), \mu(x)\}.$$

Since $xx^{-1} = e$,

$$\mu(e) \geq \mu(x).$$

Hence,

$$\mu(e) \geq \mu(x),$$

for every $x \in G$. □

Definition 2.1.4. *A fuzzy subgroup μ of a group G is said to be* improper *if μ is a constant function on G. Otherwise, μ is called a* proper fuzzy subgroup.

Definition 2.1.5. *A fuzzy subgroup μ of a group G is said to be a* fuzzy normal subgroup *of G if*

$$\mu(xy) = \mu(yx)$$

for every $x, y \in G$.

Let μ be a fuzzy normal subgroup of G. For any $t \in [0, 1]$, the set

$$\mu_t = \{(x, y) \in G \times G \mid \mu(xy^{-1}) \geq t\}$$

*is called the t-*level relation *of μ.*

For a fuzzy normal subgroup μ of G and $t \in [0, 1]$, the relation μ_t is a congruence relation on the group G.

Scanned with OKEN Scanner

Theorem 2.1.4. *Let μ be a fuzzy subgroup of a group G and $x \in G$.Then$\mu(xy)$=$\mu(y)$ for every $y \in G$ if and only if $\mu(x)$=$\mu(e)$.*

Proof. Suppose that $\mu(x) = \mu(e)$.

Since μ is a fuzzy subgroup,

$$\mu(xy) \geq \min\{\mu(x), \mu(y)\} = \min\{\mu(e), \mu(y)\} = \mu(y).$$

Again,

$$\mu(y) = \mu(x^{-1}(xy)) \geq \min\{\mu(x^{-1}), \mu(xy)\}.$$

Since

$$\mu(x^{-1}) = \mu(x) = \mu(e),$$

we obtain

$$\mu(y) \geq \mu(xy).$$

Hence,

$$\mu(xy) = \mu(y).$$

Conversely, suppose that

$$\mu(xy) = \mu(y)$$

for every $y \in G$.

Taking $y = e$, we get

$$\mu(x) = \mu(e).$$

Therefore,

$$\mu(xy) = \mu(y) \quad \text{if and only if} \quad \mu(x) = \mu(e).$$

□

Definition 2.1.6. *Let μ be a fuzzy subgroup of a group G. For any $a \in G$, the fuzzy set $a\mu$ defined by*

$$(a\mu)(x) = \mu(a^{-1}x), \quad \forall x \in G,$$

is called the fuzzy coset of G determined by a and μ.

Definition 2.1.7. *Let μ be a fuzzy subgroup of a group G. Define*

$$H = \{x \in G \mid \mu(x) = \mu(e)\},$$

where e is the identity element of G.

The order *of the fuzzy subgroup μ, denoted by $o(\mu)$, is defined as the order of the subgroup H, that is,*

$$o(\mu) = o(H).$$

Theorem 2.1.5. *Every subgroup H of a group G can be realized as a level subgroup of some fuzzy subgroup of G.*

Proof. Let H be a subgroup of G. Define a fuzzy subset

$$\mu : G \to [0, 1]$$

by

$$\mu(x) = \begin{cases} 1, & x \in H, \\ t, & x \notin H, \end{cases}$$

Scanned with OKEN Scanner

where $0 \leq t < 1$.

We show that μ is a fuzzy subgroup.

For any $x, y \in G$, if both belong to H, then

$$\mu(xy^{-1}) = 1 = \min\{\mu(x), \mu(y)\}.$$

If at least one of x or y does not belong to H, then

$$\min\{\mu(x), \mu(y)\} = t,$$

and since $\mu(xy^{-1})$ is either 1 or t, we have

$$\mu(xy^{-1}) \geq t = \min\{\mu(x), \mu(y)\}.$$

Hence μ is a fuzzy subgroup of G.

Now the level subgroup corresponding to level 1 is

$$\mu_1 = \{x \in G : \mu(x) \geq 1\} = H.$$

Therefore H is realized as a level subgroup of the fuzzy subgroup μ. □

Theorem 2.1.6. *A group G is a Dedekind group if and only if every fuzzy subgroup of G is normal.*

Proof. Suppose G is a Dedekind group. Then every subgroup of G is normal.

Let μ be any fuzzy subgroup of G.Then every level subset

$$\mu_t = \{x \in G : \mu(x) \geq t\}$$

is a subgroup of G. Since G is a Dedekind group, each level subgroup μ_t is normal.

Let $x, y \in G$. we have,

$$x \in \mu_t \iff y^{-1}xy \in \mu_t$$

for every level t. Hence

$$\mu(x) = \mu(y^{-1}xy),$$

that is μ is a normal fuzzy subgroup. Therefore every fuzzy subgroup of G is normal.

Conversely, suppose every fuzzy subgroup of G is normal. Let H be any subgroup of G. Then there exists a fuzzy subgroup μ of G such that

$$H = \mu_1.$$

Since every fuzzy subgroup is normal, μ is normal. Therefore every level subgroup of μ, in particular $\mu_1 = H$, is a normal subgroup of G.

Hence every subgroup of G is normal, and therefore G is a Dedekind group. □

Theorem 2.1.7. *Let G be a cyclic group of prime order. Then there exists a fuzzy subgroup A of G such that*

$$A(e) = t_0, \qquad A(x) = t_1 \text{ for every } x \neq e,$$

where $t_0 > t_1$.

Scanned with OKEN Scanner

Proof. Let $G = \langle a \rangle$ be a cyclic group of prime order p with identity element e.

Choose numbers $t_0, t_1 \in [0, 1]$ such that

$$t_0 > t_1.$$

Define

$$A(x) = \begin{cases} t_0, & x = e, \\ t_1, & x \neq e. \end{cases}$$

For any $x, y \in G$, the value of

$$\min\{A(x), A(y)\}$$

is either t_0 or t_1.

If $x = y = e$, then

$$A(xy^{-1}) = A(e) = t_0 = \min\{A(x), A(y)\}.$$

Otherwise,

$$A(xy^{-1}) \in \{t_0, t_1\},$$

and therefore

$$A(xy^{-1}) \geq t_1 = \min\{A(x), A(y)\}.$$

Hence

$$A(xy^{-1}) \geq \min\{A(x), A(y)\}$$

for all $x, y \in G$, so A is a fuzzy subgroup of G.

Thus there exists a fuzzy subgroup satisfying

$$A(e) = t_0, \qquad A(x) = t_1 \quad (x \neq e).$$

□

Theorem 2.1.8. *Let G be a finite group of order n and A be a fuzzy subgroup of G. Let*

$$\mathrm{Im}(A) = \{t_i \mid A(x) = t_i \text{ for some } x \in G\}.$$

Then $\{A_{t_i}\}$ are the only level subgroups of A.

Proof. Let

$$\mathrm{Im}(A) = \{t_0, t_1, \ldots, t_r\},$$

where

$$t_0 > t_1 > \cdots > t_r.$$

For each $t_i \in \mathrm{Im}(A)$, the level subset

$$A_{t_i} = \{x \in G \mid A(x) \geq t_i\}$$

is a subgroup of G.

Now let $t \in [0, A(e)]$ be any level. If $t \notin \mathrm{Im}(A)$, then there exists an index i such that

$$t_i > t > t_{i+1}.$$

Since no element of G has membership value strictly between t_i and t_{i+1},

$$A_t = \{x \in G \mid A(x) \geq t\} = \{x \in G \mid A(x) \geq t_{i+1}\} = A_{t_{i+1}}.$$

Hence every level subgroup is equal to one of the level subgroups

$$A_{t_i}, \qquad t_i \in \mathrm{Im}(A).$$

Therefore, $\{A_{t_i}\}$ are the only distinct level subgroups of A. □

Scanned with OKEN Scanner

2.2 Conjugacy of Fuzzy Subgroups

Definition 2.2.1. *Let λ and μ be two fuzzy subgroups of a group G. Then λ and μ are said to be* conjugate fuzzy subgroups *of G if there exists an element $g \in G$ such that*

$$\lambda(x) = \mu(g^{-1}xg), \qquad \forall x \in G.$$

Theorem 2.2.1. *If λ and μ are conjugate fuzzy subgroups of a group G, then*

$$o(\lambda) = o(\mu).$$

Proof. Since λ and μ are conjugate fuzzy subgroups, there exists an element $g \in G$ such that

$$\lambda(x) = \mu(g^{-1}xg), \qquad \forall x \in G.$$

Let

$$H_\lambda = \{x \in G \mid \lambda(x) = \lambda(e)\}$$

and

$$H_\mu = \{y \in G \mid \mu(y) = \mu(e)\}.$$

Define a mapping

$$\phi : H_\lambda \to H_\mu$$

by

$$\phi(x) = g^{-1}xg.$$

Let $x \in H_\lambda$. Then

$$\lambda(x) = \lambda(e).$$

Since

$$\lambda(x) = \mu(g^{-1}xg),$$

we have

$$\mu(g^{-1}xg) = \mu(e).$$

Hence,

$$g^{-1}xg \in H_\mu.$$

Therefore, ϕ is well-defined.

The mapping ϕ is one-to-one and onto since conjugation by g is a bijection. Hence,

$$|H_\lambda| = |H_\mu|.$$

By the definition of the order of a fuzzy subgroup,

$$o(\lambda) = |H_\lambda|$$

and

$$o(\mu) = |H_\mu|.$$

Therefore,

$$o(\lambda) = o(\mu).$$

Hence proved. $\square$

Scanned with OKEN Scanner

Definition 2.2.2. *Let μ be a fuzzy subgroup of a group G, and let $a, b \in G$. The* fuzzy middle coset *of μ determined by a and b is the fuzzy subset $a\mu b$ of G defined by*

$$(a\mu b)(x) = \mu(a^{-1}xb^{-1}), \qquad \forall x \in G.$$

Example 2.2.1. *Let*

$$G = \{1, -1, i, -i\}$$

be the group with respect to the usual multiplication.

Define the fuzzy subgroup

$$\mu : G \to [0, 1]$$

by

$$\mu(x) = \begin{cases} 1, & \text{if } x = 1, \\ 0.5, & \text{if } x = -1, \\ 0, & \text{if } x = i, -i. \end{cases}$$

Take

$$a = -1, \qquad b = -i.$$

The fuzzy middle coset is defined by

$$(a\mu b)(x) = \mu(a^{-1}xb^{-1}).$$

Since

$$(-1)^{-1} = -1, \qquad (-i)^{-1} = i,$$

we have

$$(a\mu b)(x) = \mu((-1)xi).$$

compute for each element of G.

$$\begin{aligned} (a\mu b)(1) &= \mu((-1)(1)i) \\ &= \mu(-i) \\ &= 0. \end{aligned}$$

$$\begin{aligned} (a\mu b)(-1) &= \mu((-1)(-1)i) \\ &= \mu(i) \\ &= 0. \end{aligned}$$

$$\begin{aligned} (a\mu b)(i) &= \mu((-1)(i)i) \\ &= \mu(1) \\ &= 1. \end{aligned}$$

$$\begin{aligned} (a\mu b)(-i) &= \mu((-1)(-i)i) \\ &= \mu(-1) \\ &= 0.5. \end{aligned}$$

Scanned with OKEN Scanner

Hence,

$$(a\mu b)(x) = \begin{cases} 1, & \text{if } x = i, \\ 0.5, & \text{if } x = -i, \\ 0, & \text{if } x = 1, -1. \end{cases}$$

Example 2.2.2. *Consider the infinite group*

$$\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$$

under the usual addition.

$$2\mathbb{Z} = \{\ldots, -4, -2, 0, 2, 4, \ldots\}$$

is a subgroup of $\mathbb{Z}$.

Define the fuzzy subgroup

$$\mu : \mathbb{Z} \to [0, 1]$$

by

$$\mu(x) = \begin{cases} 0.9, & \text{if } x \in 2\mathbb{Z}, \\ 0.8, & \text{if } x \in 2\mathbb{Z} + 1. \end{cases}$$

Let $a \in 2\mathbb{Z}$ *and* $b \in 2\mathbb{Z} + 1$.

The fuzzy middle coset is defined by

$$(a\mu b)(x) = \mu(a^{-1} + x + b^{-1}).$$

Since the operation is addition,

$$a^{-1} = -a, \qquad b^{-1} = -b.$$

Hence,

$$(a\mu b)(x) = \mu(x - a - b).$$

If, a is even and b is odd. Therefore,

$$a + b = odd.$$

Hence,

$$x - a - b = x - (odd),$$

Case 1: *Suppose* $x \in 2\mathbb{Z}$.

Then x is even, so

$$x - a - b = even - odd = odd.$$

Therefore,

$$(a\mu b)(x) = \mu(x - a - b) = 0.8.$$

Case 2: *Suppose* $x \in 2\mathbb{Z} + 1$.

Then x is odd, so

$$x - a - b = odd - odd = even.$$

Therefore,

$$(a\mu b)(x) = \mu(x - a - b) = 0.9.$$

Hence,

$$(a\mu b)(x) = \begin{cases} 0.8, & \text{if } x \in 2\mathbb{Z}, \\ 0.9, & \text{if } x \in 2\mathbb{Z} + 1. \end{cases}$$

Scanned with OKEN Scanner

Theorem 2.2.2. *Let μ be a fuzzy subgroup of a group G. Then, for every $a \in G$, the fuzzy middle coset $a\mu a^{-1}$ is also a fuzzy subgroup of G.*

Proof. The fuzzy middle coset is defined by

$$(a\mu a^{-1})(x) = \mu(a^{-1}xa), \qquad \forall x \in G.$$

To prove that $a\mu a^{-1}$ is a fuzzy subgroup, we verify the two conditions.
Let $x, y \in G$. Then

$$\begin{aligned}(a\mu a^{-1})(xy) &= \mu(a^{-1}(xy)a) \\ &= \mu(a^{-1}xya) \\ &= \mu\big((a^{-1}xa)(a^{-1}ya)\big).\end{aligned}$$

Since μ is a fuzzy subgroup,

$$\mu\big((a^{-1}xa)(a^{-1}ya)\big) \geq \min\{\mu(a^{-1}xa), \mu(a^{-1}ya)\}.$$

Therefore,

$$(a\mu a^{-1})(xy) \geq \min\{(a\mu a^{-1})(x), (a\mu a^{-1})(y)\}.$$

Now,

$$\begin{aligned}(a\mu a^{-1})(x^{-1}) &= \mu(a^{-1}x^{-1}a) \\ &= \mu\big((a^{-1}xa)^{-1}\big) \\ &= \mu(a^{-1}xa) \\ &= (a\mu a^{-1})(x),\end{aligned}$$

since $\mu(z^{-1}) = \mu(z)$ for every $z \in G$.
Hence both conditions of a fuzzy subgroup are satisfied.
Therefore, $a\mu a^{-1}$ is a fuzzy subgroup of G.
Hence proved.

Theorem 2.2.3. *Let μ be any fuzzy subgroup of a group G and let $a\mu a^{-1}$ be a fuzzy middle coset of the group G. Then*

$$o(a\mu a^{-1}) = o(\mu)$$

for any $a \in G$.

Proof. Let μ be a fuzzy subgroup of a group G and let $a \in G$. By Theorem 1.2.10, the fuzzy middle coset $a\mu a^{-1}$ is a fuzzy subgroup of the group G. Further, by the definition of a fuzzy middle coset of the group G, we have

$$(a\mu a^{-1})(x) = \mu(a^{-1}xa)$$

for every $x \in G$.

Scanned with OKEN Scanner

Hence, for any $a \in G$, μ and $a\mu a^{-1}$ are conjugate fuzzy subgroups of the group G, since there exists $a \in G$ such that

$$(a\mu a^{-1})(x) = \mu(a^{-1}xa)$$

for every $x \in G$.

By using the earlier theorem, which states that the orders of conjugate fuzzy subgroups are equal, we obtain

$$o(a\mu a^{-1}) = o(\mu)$$

for any $a \in G$. □

Example 2.2.3. *Let $G = S_3$ be the symmetric group of degree 3. Choose $p_1, p_2, p_3 \in [0, 1]$ such that*

$$p_1 \geq p_2 \geq p_3.$$

Define the fuzzy subgroup $\mu : G \to [0, 1]$ by

$$\mu(x) = \begin{cases} p_1, & \text{if } x = e, \\ p_2, & \text{if } x = (12), \\ p_3, & \text{otherwise.} \end{cases}$$

Since only the identity element has the highest membership value,

$$\{x \in G \mid \mu(x) = \mu(e)\} = \{e\}.$$

Hence,

$$o(\mu) = 1.$$

Now consider the fuzzy middle coset

$$(a\mu a^{-1})(x) = \mu(a^{-1}xa), \qquad a \in G.$$

We examine different choices of a.

- *If $a = e$, then*

  $$a\mu a^{-1} = \mu,$$

  so

  $$o(a\mu a^{-1}) = 1.$$

- *If $a = (12)$, then (12) remains fixed under conjugation. Hence*

  $$(a\mu a^{-1})(x) = \begin{cases} p_1, & x = e, \\ p_2, & x = (12), \\ p_3, & \text{otherwise.} \end{cases}$$

  Therefore,

  $$o(a\mu a^{-1}) = 1.$$

Scanned with OKEN Scanner

- *If $a = (13)$ or $a = (132)$, then conjugation sends (12) to (23). Thus*

$$(a\mu a^{-1})(x) = \begin{cases} p_1, & x = e, \\ p_2, & x = (23), \\ p_3, & \textit{otherwise}. \end{cases}$$

Hence,

$$o(a\mu a^{-1}) = 1.$$

- *If $a = (23)$ or $a = (123)$, then conjugation sends (12) to (13). Therefore*

$$(a\mu a^{-1})(x) = \begin{cases} p_1, & x = e, \\ p_2, & x = (13), \\ p_3, & \textit{otherwise}. \end{cases}$$

Again,

$$o(a\mu a^{-1}) = 1.$$

Thus, for every $a \in G$,

$$o(a\mu a^{-1}) = o(\mu) = 1.$$

Although the fuzzy subgroups μ and $a\mu a^{-1}$ are not always identical, they have the same order.

Example 2.2.4. *Let $G = S_3$ be the symmetric group of degree 3. Choose $p_1, p_2, p_3 \in [0, 1]$ such that*

$$p_1 \geq p_2 \geq p_3.$$

Define the fuzzy subgroup $\mu : G \to [0, 1]$ by

$$\mu(x) = \begin{cases} p_1, & \textit{if } x = e, \\ p_2, & \textit{if } x = (123) \textit{ or } (132), \\ p_3, & \textit{otherwise}. \end{cases}$$

Clearly, μ is a fuzzy subgroup of G.
Now consider

$$(a\mu a^{-1})(x) = \mu(a^{-1}xa), \qquad a \in G.$$

Since the two 3-cycles (123) and (132) are conjugate to each other, conjugation by any element of S_3 simply permutes these two elements. Hence,

$$(a\mu a^{-1})(x) = \begin{cases} p_1, & \textit{if } x = e, \\ p_2, & \textit{if } x = (123) \textit{ or } (132), \\ p_3, & \textit{otherwise}. \end{cases}$$

Therefore,

$$(a\mu a^{-1})(x) = \mu(x)$$

for every $x \in G$ and every $a \in G$.

Scanned with OKEN Scanner

Thus,

$$a\mu a^{-1} = \mu$$

for all $a \in G$.

Hence, this example shows that even though S_3 is a non-abelian group, it is still possible to have

$$a\mu a^{-1} = \mu$$

for every $a \in G$.

Theorem 2.2.4. *Let μ be a fuzzy subgroup of a finite group G. Then*

$$o(\mu) \mid o(G).$$

Proof. Let μ be a fuzzy subgroup of the finite group G, and let e be the identity element of G.

Consider the set

$$H = \{x \in G \mid \mu(x) = \mu(e)\}.$$

Since H is the $\mu(e)$-level subgroup of μ, it follows that H is a subgroup of G.

By the definition of the order of a fuzzy subgroup,

$$o(\mu) = o(H).$$

Now, by Lagrange's Theorem,

$$o(H) \mid o(G).$$

Hence,

$$o(\mu) \mid o(G).$$

Therefore, the order of every fuzzy subgroup of a finite group divides the order of the group. □

Theorem 2.2.5. *Let λ and μ be two improper fuzzy subgroups of a group G. Then λ and μ are conjugate fuzzy subgroups of G if and only if*

$$\lambda = \mu.$$

Proof. Suppose λ and μ are conjugate fuzzy subgroups of G. Then there exists $g \in G$ such that

$$\lambda(x) = \mu(g^{-1}xg)$$

for every $x \in G$.

Since μ is an improper fuzzy subgroup, it is constant on G. Hence,

$$\mu(g^{-1}xg) = \mu(x)$$

for every $x \in G$.

Therefore,

$$\lambda(x) = \mu(x)$$

for every $x \in G$, which implies

$$\lambda = \mu.$$

Scanned with OKEN Scanner

Conversely, suppose that $\lambda = \mu$. Let e be the identity element of G. Then, for every $x \in G$,

$$\lambda(x) = \mu(x) = \mu(e^{-1}xe).$$

Hence, there exists an element $e \in G$ such that

$$\lambda(x) = \mu(e^{-1}xe)$$

for every $x \in G$. Therefore, λ and μ are conjugate fuzzy subgroups of G.

Hence,

$$\lambda \text{ and } \mu \text{ are conjugate} \iff \lambda = \mu.$$

$\square$

Theorem 2.2.6. *Let λ and μ be two fuzzy subsets of an abelian group G. Then λ and μ are conjugate fuzzy subsets of the group G if and only if*

$$\lambda = \mu.$$

Proof. Assume that λ and μ are conjugate fuzzy subsets of the group G. Then there exists some $g \in G$ such that

$$\lambda(x) = \mu(g^{-1}xg) \qquad \text{for every } x \in G.$$

Since G is an abelian group, we have

$$g^{-1}xg = g^{-1}gx = x.$$

Hence,

$$\lambda(x) = \mu(g^{-1}xg) = \mu(g^{-1}gx) = \mu(x)$$

for every $x \in G$.

Therefore,

$$\lambda = \mu.$$

Conversely, suppose that $\lambda = \mu$. Let e be the identity element of the group G. Then

$$\lambda(x) = \mu(x) = \mu(e^{-1}xe)$$

for every $x \in G$.

Thus, by the definition of conjugate fuzzy subsets, λ and μ are conjugate fuzzy subsets of the group G.

Hence,

$$\lambda \text{ and } \mu \text{ are conjugate fuzzy subsets of } G \iff \lambda = \mu.$$

$\square$

$\lambda(x) = \mu(x) = \mu(e^{-1}xe)$ for every $x \in G$

Example 2.2.5. *Let $G = S_3$ and let $\lambda, \mu : S_3 \to [0,1]$ be defined by*

$$\lambda(x) = \begin{cases} 0.5, & \textit{if } x = e, \\ 0.4, & \textit{if } x = (123) \textit{ or } x = (132), \\ 0.3, & \textit{otherwise}, \end{cases}$$

Scanned with OKEN Scanner

and

$$\mu(x) = \begin{cases} 0.6, & \textit{if } x = e, \\ 0.5, & \textit{if } x = (23), \\ 0.3, & \textit{otherwise.} \end{cases}$$

We claim that λ and μ are not conjugate fuzzy subsets of the symmetric group S_3. the identity element must remain fixed under conjugation. Hence, if λ and μ were conjugate, then there would exist some $g \in S_3$ such that

$$\lambda(e) = \mu(g^{-1}eg) = \mu(e).$$

But

$$\lambda(e) = 0.5 \neq 0.6 = \mu(e).$$

This is a contradiction.

Therefore, λ and μ are not conjugate fuzzy subsets of S_3.

2.3 Fuzzy Relations

Definition 2.3.1. *Let R_λ and R_μ be two fuzzy relations on a group G. Then R_λ and R_μ are said to be* conjugate fuzzy relations *on G if there exists $(g_1, g_2) \in G \times G$ such that*

$$R_\lambda(x, y) = R_\mu\big(g_1^{-1}xg_1,\ g_2^{-1}yg_2\big)$$

for every $(x, y) \in G \times G$.

Definition 2.3.2. *Let R_λ and R_μ be two fuzzy relations on a group G. They are called* generalized conjugate fuzzy relations *on G if there exists an element $g \in G$ such that*

$$R_\lambda(x, y) = R_\mu\big(g^{-1}xg,\ g^{-1}yg\big)$$

for every $(x, y) \in G \times G$.

Theorem 2.3.1. *Let R_λ and R_μ be two fuzzy relations on a group G. If R_λ and R_μ are generalized conjugate fuzzy relations on G, then R_λ and R_μ are conjugate fuzzy relations on G.*

Proof. Suppose R_λ and R_μ are generalized conjugate fuzzy relations on the group G. Then there exists an element $g \in G$ such that

$$R_\lambda(x, y) = R_\mu\big(g^{-1}xg,\ g^{-1}yg\big)$$

for every $(x, y) \in G \times G$.

Now choose

$$g_1 = g_2 = g.$$

Then $(g_1, g_2) \in G \times G$, and hence

$$g_1^{-1}xg_1 = g^{-1}xg \quad \text{and} \quad g_2^{-1}yg_2 = g^{-1}yg.$$

Therefore,

$$R_\lambda(x, y) = R_\mu\big(g_1^{-1}xg_1,\ g_2^{-1}yg_2\big)$$

for every $(x, y) \in G \times G$.

This satisfies the definition of conjugate fuzzy relations. Hence R_λ and R_μ are conjugate fuzzy relations on the group G. □

Scanned with OKEN Scanner

Theorem 2.3.2. *Let R_λ and R_μ be generalized conjugate fuzzy relations on a group G. If R_λ is a similarity relation on G, then R_μ is also a similarity relation on G.*

Proof. Since R_λ and R_μ are generalized conjugate fuzzy relations, there exists $g \in G$ such that

$$R_\lambda(x, y) = R_\mu(g^{-1}xg,\ g^{-1}yg)$$

for every $x, y \in G$.

Since R_λ is a similarity relation, it is reflexive, symmetric and transitive.

For reflexivity, let $u \in G$. Then

$$R_\mu(u, u) = R_\lambda(gug^{-1}, gug^{-1}) = 1.$$

Hence R_μ is reflexive.

Next, let $u, v \in G$. Since R_λ is symmetric,

$$R_\mu(u, v) = R_\lambda(gug^{-1}, gvg^{-1}) = R_\lambda(gvg^{-1}, gug^{-1}) = R_\mu(v, u).$$

Thus R_μ is symmetric.

Finally, let $u, v, w \in G$. Since R_λ is transitive,

$$\min\{R_\lambda(gug^{-1}, gvg^{-1}), R_\lambda(gvg^{-1}, gwg^{-1})\} \le R_\lambda(gug^{-1}, gwg^{-1}).$$

Using the generalized conjugacy relation, we obtain

$$\min\{R_\mu(u, v), R_\mu(v, w)\} \le R_\mu(u, w).$$

Hence R_μ is transitive.

Therefore, R_μ is reflexive, symmetric and transitive. Hence R_μ is a similarity relation on G. □

Definition 2.3.3. *Let S_n denote the symmetric group on the set $\{1, 2, \ldots, n\}$. We define the following:*

1. *Let $F(S_n)$ denote the collection of all fuzzy subgroups of S_n.*
2. *If $f \in F(S_n)$, then the image of f is*
$$\operatorname{Im}(f) = \{\, f(x) \mid x \in S_n \,\}.$$
3. *Let $f, g \in F(S_n)$. If*
$$|\operatorname{Im}(f)| < |\operatorname{Im}(g)|,$$
then we write $f < g$. This relation is used to define the maximal element of $F(S_n)$.
4. *A fuzzy subgroup f of S_n is called a* fuzzy symmetric subgroup *of S_n if*
$$f = \max F(S_n).$$

Theorem 2.3.3. *Let f be a fuzzy symmetric subgroup of the symmetric group S_3. Then*

$$o(\operatorname{Im} f) = 3.$$

Scanned with OKEN Scanner

Proof. Since f is a fuzzy symmetric subgroup of S_3, we have

$$f = \max F(S_3),$$

where $F(S_3)$ denotes the set of all fuzzy subgroups of S_3.

By Theorem 2.1.8, the distinct level subgroups of a fuzzy subgroup are in one-to-one correspondence with the distinct elements of $\mathrm{Im}(f)$. Hence

$$o(\mathrm{Im}\, f)$$

is equal to the number of distinct level subgroups of f.

The subgroup chain of maximum length in S_3 is

$$\{e\} \subset \langle(123)\rangle \subset S_3.$$

That is there are exactly three distinct level subgroups corresponding to a fuzzy symmetric subgroup of S_3.

Therefore,

$$o(\mathrm{Im}\, f) = 3.$$

□

Scanned with OKEN Scanner

Chapter 3

The Number of Fuzzy Subgroups of Finite Cyclic Groups

Finite cyclic groups play a significant role in group theory due to their simple structure and well-understood subgroup lattice. The correspondence between fuzzy subgroups and chains of ordinary subgroups makes it possible to determine the number of distinct fuzzy subgroups of finite cyclic groups using lattice-theoretic methods.

Now we study the number of fuzzy subgroups of finite cyclic groups and discuss the techniques used to enumerate them through subgroup chains and equivalence classes.

3.1 Characterisation of Fuzzy Subgroups

Theorem 3.1.1. *A fuzzy subset μ of a group G is a fuzzy subgroup if and only if there exists a chain of subgroups*

$$P_1(\mu) \leq P_2(\mu) \leq \cdots \leq P_n(\mu) = G$$

such that μ takes a constant value on each level of the chain. That is,

$$\mu(x) = \begin{cases} \theta_1, & x \in P_1(\mu), \\ \theta_2, & x \in P_2(\mu) \setminus P_1(\mu), \\ \vdots & \vdots \\ \theta_n, & x \in G \setminus P_{n-1}(\mu), \end{cases}$$

where

$$\theta_1 > \theta_2 > \cdots > \theta_n.$$

Proof. Assume first that μ is a fuzzy subgroup of G. Since G is finite, the membership values of μ are finite in number. Let these values be

$$\theta_1 > \theta_2 > \cdots > \theta_n.$$

For each θ_i, define

$$P_i(\mu) = \{x \in G : \mu(x) \geq \theta_i\}.$$

Scanned with OKEN Scanner

Since μ is a fuzzy subgroup, every level subset $P_i(\mu)$ is a subgroup of G. Moreover,

$$P_1(\mu) \leq P_2(\mu) \leq \cdots \leq P_n(\mu) = G.$$

Also, every element of G belongs to exactly one layer of this chain, so μ remains constant on each layer with corresponding membership value θ_i. Hence μ has the required representation.

Conversely, suppose there exists a chain of subgroups

$$P_1(\mu) \leq P_2(\mu) \leq \cdots \leq P_n(\mu) = G$$

and μ is defined by assigning the constants

$$\theta_1 > \theta_2 > \cdots > \theta_n$$

to the successive layers of the chain. Then every level subset of μ is one of the subgroups in the above chain. Therefore each level subset is a subgroup of G. Hence, by the level subgroup characterization of fuzzy subgroups, μ is a fuzzy subgroup of G.

Thus the result follows. $\square$

Example 3.1.1. *Consider the group $G = \mathbb{Z}_{24}$. Define the functions μ, γ, α and β as follows:*

$$\mu(x) = \begin{cases} 1, & x \in \{0, 4, 8, 12, 14, 16, 18, 20, 22\}, \\ \frac{1}{2}, & x \in \{1, 2, 3, 5, 6, 7, 9, 10, 11, 13, 15, 17, 19, 21, 23\}. \end{cases}$$

$$\gamma(x) = \begin{cases} 1, & x \in \{0, 6, 12, 18\}, \\ \frac{1}{2}, & x \in \{2, 4, 8, 10, 14, 16, 20, 22\}, \\ \frac{1}{4}, & x \in \{1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23\}. \end{cases}$$

$$\alpha(x) = \begin{cases} 0, & x \in \{0, 4, 8, 12, 16, 20\}, \\ \frac{1}{2}, & x \in \{2, 6, 10, 14, 18, 22\}, \\ \frac{1}{3}, & x \in \{1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23\}. \end{cases}$$

$$\beta(x) = \begin{cases} 1, & x \in \{0, 4, 8, 12, 16, 20\}, \\ \frac{1}{2}, & x \in \{2, 6, 10, 14, 18, 22\}, \\ \frac{1}{4}, & x \in \{1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23\}. \end{cases}$$

Note that

$$P_1(\mu) = \{0, 4, 8, 12, 14, 16, 18, 20, 22\}$$

is not a subgroup of $\mathbb{Z}_{24}$. According to Theorem 3.1.1, μ is not a fuzzy subgroup of $\mathbb{Z}_{24}$.

Also, $P_1(\gamma)$, $P_2(\gamma)$ and $P_3(\gamma)$ are subgroups of $\mathbb{Z}_{24}$ with

$$P_3(\gamma) = G.$$

Hence, by Theorem 3.1.1, γ is a fuzzy subgroup of $\mathbb{Z}_{24}$.

Similarly, we can show that α and β are fuzzy subgroups of $\mathbb{Z}_{24}$.

Scanned with OKEN Scanner

3.2 Equivalence of Fuzzy Subgroups

Definition 3.2.1. *Let μ and γ be fuzzy subgroups of G of the form*

$$\mu(x) = \begin{cases} \theta_1, & x \in A_1, \\ \theta_2, & x \in A_2, \\ \vdots & \vdots \\ \theta_m, & x \in A_m, \end{cases} \qquad \gamma(x) = \begin{cases} \delta_1, & x \in B_1, \\ \delta_2, & x \in B_2, \\ \vdots & \vdots \\ \delta_n, & x \in B_n, \end{cases}$$

where $\theta_i, \delta_j \in [0,1]$ with $\theta_k > \theta_l$ and $\delta_k > \delta_l$ whenever $k < l$, and

$$\bigcup_{i=1}^{m} A_i = G, \qquad \bigcup_{j=1}^{n} B_j = G.$$

Then μ and γ are said to be equivalent *if*

$$m = n$$

and

$$A_i = B_i, \qquad \forall\, i = 1, 2, \ldots, n.$$

Example 3.2.1. *Let γ, α and β be the fuzzy subgroups defined in Example 3.1.1.*
Since

$$P_1(\gamma) \neq P_1(\alpha),$$

it follows from Definition 3.1 that γ is not equivalent to α. Similarly, γ and β are not equivalent.
Observe that

$$|\alpha| = |\beta|$$

and

$$P_i(\alpha) = P_i(\beta), \qquad i = 1, 2, 3.$$

hence,

$$\alpha \sim \beta.$$

Lemma 3.2.1. *The number of fuzzy subgroups of a finite group G is equal to the number of chains in the lattice subgroup of G.*

Proof. The result follows directly from Theorem 3.1.1 and Definition 3.2.1. □

Scanned with OKEN Scanner

3.3 Enumeration in cyclic groups

Let $o(F_G)$ denote the number of fuzzy subgroups of G, and let $o(F_{P_1=H})$ denote the number of fuzzy subgroups μ of G with $P_1(\mu) = H$.

$$o(F_G) = \sum_{H \leq G} o(F_{P_1=H}). \tag{3.1}$$

also,

$$o(F_{P_1=G}) = 1. \tag{3.2}$$

Theorem 3.3.1. *Let H be a subgroup of G, and let the set of all subgroups of G which contain H (but are not equal to H) be*

$$\{H_1, H_2, \ldots, H_n = G\}.$$

then,

$$o(F_{P_1=H}) = \sum_{1 \leq i \leq n} o(F_{P_1=H_i}).$$

Proof. We have

$$o(F_{P_1=H}) = \sum_{1 \leq i \leq n} o(F_{P_1=H;\, P_2=H_i}).$$

For every $i = 1, 2, \ldots, n$,

$$o(F_{P_1=H;\, P_2=H_i}) = o(F_{P_1=H_i}).$$

hence,

$$o(F_{P_1=H}) = \sum_{1 \leq i \leq n} o(F_{P_1=H_i}).$$

□

Theorem 3.3.2. *Let e be the identity element of a group G. Then*

$$o(F_G) = 2\, o(F_{P_1=\{e\}}).$$

Proof. From equation (3.1), we have

$$o(F_G) = \sum_{H \leq G} o(F_{P_1=H}).$$

therefore,

$$o(F_G) = o(F_{P_1=\{e\}}) + \sum_{\{e\} \neq H \leq G} o(F_{P_1=H}).$$

$$\sum_{\{e\} \neq H \leq G} o(F_{P_1=H}) = o(F_{P_1=\{e\}}).$$

hence,

$$o(F_G) = o(F_{P_1=\{e\}}) + o(F_{P_1=\{e\}}) = 2\, o(F_{P_1=\{e\}}).$$

□

Scanned with OKEN Scanner

Theorem 3.3.3. *Let $G = Z_p \times Z_q$, where p and q are distinct prime numbers. Then the number of fuzzy subgroups of G is 6.*

Proof. The order of G is pq. Hence, every nontrivial subgroup of G must be of order p or q. Therefore, G has only two nontrivial subgroups, namely

$$H_1 = Z_p \times \{0\} \quad \text{and} \quad H_2 = \{0\} \times Z_q.$$

The subgroup lattice of G is shown in Figure 3.1

$$\begin{tikzcd}
 & G \arrow[dl, dash] \arrow[dr, dash] & \\
H_1 = Z_p \times \{0\} \arrow[dr, dash] & & H_2 = \{0\} \times Z_q \arrow[dl, dash] \\
 & \{e\} &
\end{tikzcd}$$

Figure 3.1: Subgroup lattice of $G = Z_p \times Z_q$.

According to equation (3.2),

$$o(F_{P_1=G}) = 1.$$

$$o(F_{P_1=H_1}) = o(F_{P_1=H_2}) = 1,$$

and

$$o(F_{P_1=\{e\}}) = 3.$$

$$o(F_G) = 2\, o(F_{P_1=\{e\}}) = 2 \times 3 = 6.$$

All the fuzzy subgroups are

$$\mu_1(x) = \theta_1, \qquad \forall x \in G,$$

$$\mu_2(x) = \begin{cases} \theta_1, & x \in H_1, \\ \theta_2, & x \in G \setminus H_1, \end{cases}$$

$$\mu_3(x) = \begin{cases} \theta_1, & x \in H_2, \\ \theta_2, & x \in G \setminus H_2, \end{cases}$$

$$\mu_4(x) = \begin{cases} \theta_1, & x \in \{e\}, \\ \theta_2, & x \in G \setminus \{e\}, \end{cases}$$

$$\mu_5(x) = \begin{cases} \theta_1, & x \in \{e\}, \\ \theta_2, & x \in H_1 \setminus \{e\}, \\ \theta_3, & x \in G \setminus H_1, \end{cases}$$

Scanned with OKEN Scanner

$$\mu_6(x) = \begin{cases} \theta_1, & x \in \{e\}, \\ \theta_2, & x \in H_2 \setminus \{e\}, \\ \theta_3, & x \in G \setminus H_2. \end{cases}$$

hence, the number of fuzzy subgroups of $G = Z_p \times Z_q$ is 6. □

Theorem 3.3.4. *Let $G = Z_p \times Z_q \times Z_r$, where p, q and r are distinct prime numbers. Then the number of fuzzy subgroups of G is 26.*

Proof. The order of G is pqr. Hence, every nontrivial subgroup of G must be of order p, q, r, pq, pr or qr. We have six nontrivial subgroups of G, namely

$$H_1 = Z_p \times \{0\} \times \{0\}, \qquad H_2 = \{0\} \times Z_q \times \{0\},$$

$$H_3 = \{0\} \times \{0\} \times Z_r,$$
$$H_4 = Z_p \times Z_q \times \{0\},$$
$$H_5 = Z_p \times \{0\} \times Z_r,$$
$$H_6 = \{0\} \times Z_q \times Z_r.$$

The subgroup lattice of G is shown in Figure 3.2.

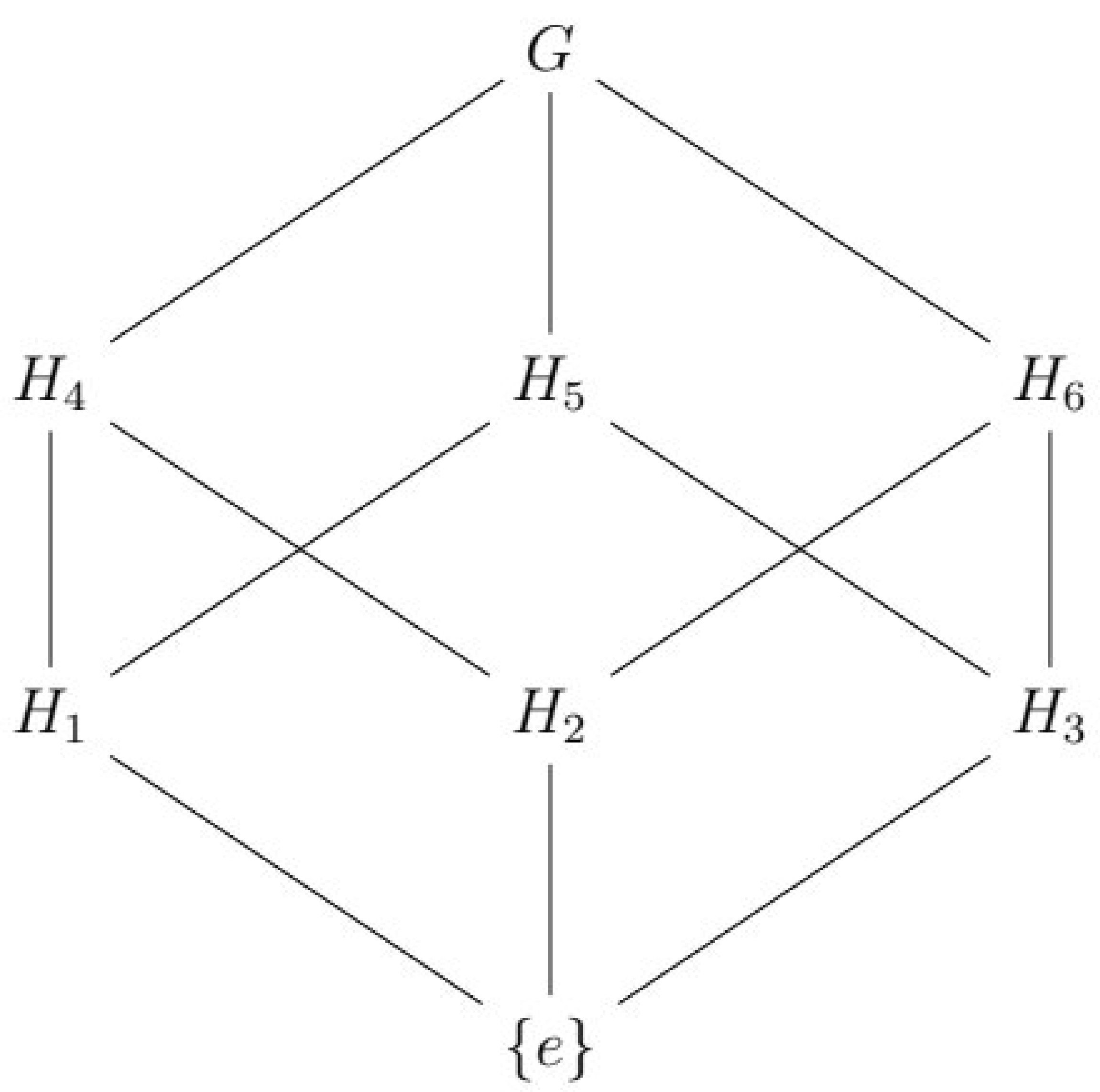

Figure 3.2: Subgroup lattice of $G = Z_p \times Z_q \times Z_r$.

According to equation (3.2), we have

$$o(F_{P_1=G}) = 1.$$

Using Theorem 3.2.1, we have

$$o(F_{P_1=H_4}) = o(F_{P_1=H_5}) = o(F_{P_1=H_6}) = 1,$$
$$o(F_{P_1=H_1}) = o(F_{P_1=H_2}) = o(F_{P_1=H_3}) = 3,$$

and

$$o(F_{P_1=\{e\}}) = 13.$$

Scanned with OKEN Scanner

Then, by Theorem 3.2.2,

$$o(F_G) = 2\,o(F_{P_1=\{e\}}) = 2 \times 13 = 26.$$

$\square$

Theorem 3.3.5. *Let $G = Z_p \times Z_q \times Z_r \times Z_s$, where p, q, r and s are distinct prime numbers. Then the number of fuzzy subgroups of G is* 142.

Proof. The order of G is $pqrs$. Hence, every nontrivial subgroup of G must be of order p, q, r, s, pq, pr, ps, qr, qs, rs, pqr, pqs, prs or qrs.

Every subgroup of G can be obtained from $G = Z_p \times Z_q \times Z_r \times Z_s$ by changing k components of Z_p, Z_q, Z_r and Z_s to $\{0\}$, where $0 \leq k \leq 4$. Note that if $k = 0$, the subgroup is G, and if $k = 4$, the subgroup is

$$\{e\} = \{0\} \times \{0\} \times \{0\} \times \{0\}.$$

We have fourteen nontrivial subgroups of G, namely,

$$H_1 = Z_p \times Z_q \times Z_r \times \{0\},$$

$$H_2 = Z_p \times Z_q \times \{0\} \times Z_s,$$

$$H_3 = Z_p \times \{0\} \times Z_r \times Z_s,$$

$$H_4 = \{0\} \times Z_q \times Z_r \times Z_s,$$

$$H_5 = Z_p \times Z_q \times \{0\} \times \{0\},$$

$$H_6 = Z_p \times \{0\} \times Z_r \times \{0\},$$

$$H_7 = Z_p \times \{0\} \times \{0\} \times Z_s,$$

$$H_8 = Z_s \times Z_q \times Z_r \times \{0\},$$

$$H_9 = \{0\} \times Z_q \times \{0\} \times Z_s,$$

$$H_{10} = \{0\} \times \{0\} \times Z_r \times Z_s,$$

$$H_{11} = Z_p \times \{0\} \times \{0\} \times \{0\},$$

$$H_{12} = \{0\} \times Z_q \times \{0\} \times \{0\},$$

$$H_{13} = \{0\} \times \{0\} \times Z_r \times \{0\},$$

Scanned with OKEN Scanner

$$H_{14} = \{0\} \times \{0\} \times \{0\} \times Z_s.$$

According to the sub group lattice, we have

$$o(F_{P_1=G}) = 1,$$

and using Theorem 3.2.1, we have

$$o(F_{P_1=H_1}) = o(F_{P_1=H_2}) = o(F_{P_1=H_3}) = o(F_{P_1=H_4}) = 1,$$

$$o(F_{P_1=H_5}) = o(F_{P_1=H_6}) = o(F_{P_1=H_7}) = o(F_{P_1=H_8}) = o(F_{P_1=H_9}) = o(F_{P_1=H_{10}}) = 3,$$

$$o(F_{P_1=H_{11}}) = o(F_{P_1=H_{12}}) = o(F_{P_1=H_{13}}) = o(F_{P_1=H_{14}}) = 12,$$

and

$$o(F_{P_1=\{e\}}) = 71.$$

Then, by Theorem 3.2.2, we get

$$o(F_G) = 2\,o(F_{P_1=\{e\}}) = 2 \times 71 = 142.$$

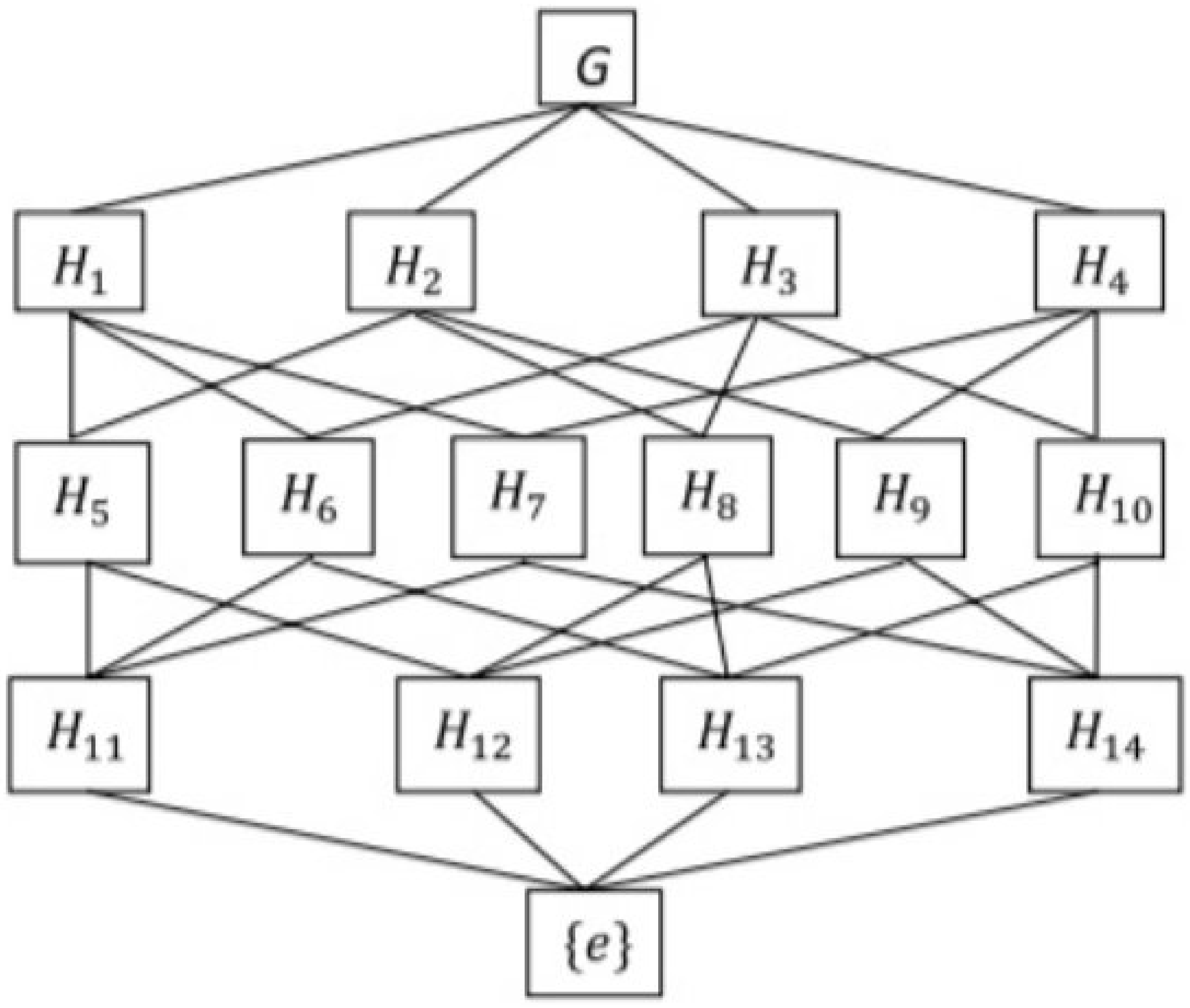

Figure 3.3: Subgroup lattice of $G = Z_p \times Z_q \times Z_r \times Z_s$

□

Scanned with OKEN Scanner

Chapter 4

The Number of Fuzzy Subgroups of Finite Symmetric Groups

In this chapter, we study the number of distinct fuzzy subgroups of finite symmetric groups. First we will discuss the relation between fuzzy subgroups and subgroup chains, which forms the basis for the enumeration process. We then determine the number of fuzzy subgroups for the symmetric groups S_3 and S_4, and discuss lower bounds for the number of fuzzy subgroups of S_n for $n \geq 5$.

4.1 Equivalence and Subgroup Chains

Definition 4.1.1. *Let G be a finite group and let $\mu, \eta \in F(G)$ be two fuzzy subgroups of G. Where, $F(G)$ is the collection of all fuzzy subsets of G. An equivalence relation $\sim$ on $F(G)$ is defined by*

$$\mu \sim \eta \iff \big(\mu(x) > \mu(y) \iff \eta(x) > \eta(y)\big), \quad \forall\, x, y \in G.$$

Two fuzzy subgroups μ and η are said to be equivalent *if $\mu \sim \eta$; otherwise, they are called* distinct.

Theorem 4.1.1. *Let G be a finite group and let $\mu, \eta \in F(G)$ be two fuzzy subgroups of G. Then*

$$\mu \sim \eta$$

if and only if μ and η have the same set of level subgroups of the form

$$\mu_{\alpha_1} \subset \mu_{\alpha_2} \subset \cdots \subset \mu_{\alpha_r} = G,$$

where

$$\mu(G) = \{\alpha_1, \alpha_2, \ldots, \alpha_r\}$$

with

$$\alpha_1 > \alpha_2 > \cdots > \alpha_r.$$

Corollary 4.1.1. *There exists a one-to-one correspondence between the equivalence classes of fuzzy subgroups of G and the set of chains of subgroups of G terminating in G. Therefore, the problem of finding the number of distinct fuzzy subgroups of a finite group is equivalent to counting the chains of subgroups that end in G.*

Scanned with OKEN Scanner

Result 4.1.1. *Let n be the order of the finite cyclic group,Let $n = p_1^{m_1} p_2^{m_2} \cdots p_s^{m_s}$ be the prime factorization of n. Then the number of distinct fuzzy subgroups of the cyclic group $\mathbb{Z}_n$ is denoted by $f(n)$ and is given by*

$$f(n) = 2^{\sum_{\alpha=1}^{s} m_\alpha} \sum_{i_2=1}^{m_2} \sum_{i_2=0}^{m_2} \sum_{i_3=0}^{m_3} \cdots \sum_{i_s=0}^{m_s} \left(-\frac{1}{2}\right)^{\sum_{\alpha=2}^{s} i_\alpha} \prod_{\alpha=2}^{s} \binom{i_{\alpha-1}}{i_\alpha} \binom{m_\alpha + \sum_{\beta=2}^{\alpha} (m_\beta - i_\beta)}{m_\alpha}.$$

When $s = 1$, the iterated sums are taken to be equal to 1.

Definition 4.1.2. *Let G be a group and let H be a proper subgroup of G. Then H is called a maximal subgroup of G if there exists no subgroup K of G such that*

$$H < K < G.$$

Definition 4.1.3. *A Maximal chain of subgroups of a group G is a chain of subgroups*

$$H_0 < H_1 < \cdots < H_n = G$$

such that each H_i is a maximal subgroup of H_{i+1} for $i = 0, 1, \ldots, n-1$. The number of maximal chains of subgroups of G is denoted by $g(G)$. The number of chains of subgroups terminating at G is denoted by $h(G)$.

Remark 4.1.1. *Let $M_1, M_2, \ldots, M_k$ be the maximal subgroups of G. Since every maximal chain of subgroups of G contains exactly one maximal subgroup, therefore*

$$g(G) = \sum_{i=1}^{k} g(M_i). \tag{4.1}$$

Result 4.1.2. *In the case of finite cyclic group,*

$$g(\mathbb{Z}_n) = \binom{m_1 + m_2 + \cdots + m_s}{m_1, m_2, \ldots, m_s} = \frac{(m_1 + m_2 + \cdots + m_s)!}{m_1!\, m_2! \cdots m_s!},$$

where

$$n = p_1^{m_1} p_2^{m_2} \cdots p_s^{m_s}.$$

Then, the number of maximal chains of subgroups of finite dihedral group D_{2n} is given by,

$$g(D_{2n}) = \sum_{\alpha_1=0}^{m_1} \sum_{\alpha_2=0}^{m_2} \cdots \sum_{\alpha_s=0}^{m_s} \binom{\alpha_1 + \alpha_2 + \cdots + \alpha_s}{\alpha_1, \alpha_2, \ldots, \alpha_s}$$
$$\binom{m_1 + m_2 + \cdots + m_s - \alpha_1 - \alpha_2 - \cdots - \alpha_s}{m_1 - \alpha_1, m_2 - \alpha_2, \ldots, m_s - \alpha_s} p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s}.$$

In particular, we get $g(D_4) = 3, g(D_6) = 4$ and $g(D_8) = 7$

Theorem 4.1.2 (Inclusion–Exclusion Principle)**.** *Let $A_1, A_2, \ldots, A_n$ be finite sets. Then*

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{i=1}^{n} |A_i| - \sum_{1 \le i < j \le n} |A_i \cap A_j| + \cdots + (-1)^{n-1} \left|\bigcap_{i=1}^{n} A_i\right|.$$

Scanned with OKEN Scanner

4.2 Recurrence Relation for $h(G)$

In this section, we derive a recurrence relation for the number of subgroup chains terminating at a finite group G. Let $M_1, M_2, \ldots, M_k$ be the maximal subgroups of G.

Let

$$C = \{\, H_1 \subset H_2 \subset \cdots \subset H_r = G \,\}$$

be the set of all subgroup chains ending in G. Also, let

$$C' = \{\, H_1 \subset H_2 \subset \cdots \subset H_r \neq G \,\}$$

be the set of all subgroup chains which do not terminate at G.

Clearly,

$$|C| = 1 + |C'|,$$

since the chain consisting only of the group G is counted separately.

For each maximal subgroup M_i, $i = 1, 2, \ldots, k$, define

$$C_i = \{\textit{all chains of } C' \textit{ contained in } M_i\}.$$

Since every proper subgroup of G is contained in at least one maximal subgroup, every chain in C' belongs to one of the sets C_i. Hence,

$$C' = \bigcup_{i=1}^{k} C_i.$$

Applying the Inclusion–Exclusion Principle,

$$\begin{aligned} |C| &= 1 + \left| \bigcup_{i=1}^{k} C_i \right| \\ &= 1 + \sum_{i=1}^{k} |C_i| - \sum_{1 \le i_1 < i_2 \le k} |C_{i_1} \cap C_{i_2}| \\ &\quad + \cdots + (-1)^{k-1} \left| \bigcap_{i=1}^{k} C_i \right|. \end{aligned}$$

Observe that for any $1 \le r \le k$ and $1 \le i_1 < i_2 < \cdots < i_r \le k$,

$$C_{i_1} \cap C_{i_2} \cap \cdots \cap C_{i_r}$$

is precisely the collection of subgroup chains contained in

$$M_{i_1} \cap M_{i_2} \cap \cdots \cap M_{i_r}.$$

Therefore,

$$|C_{i_1} \cap \cdots \cap C_{i_r}| = 2h(M_{i_1} \cap \cdots \cap M_{i_r}) - 1.$$

Substituting these expressions into the above equation, we obtain

Scanned with OKEN Scanner

$$\begin{aligned}|C| = 1 + &\sum_{i=1}^{k}(2h(M_i) - 1) \\ &- \sum_{1\le i_1<i_2\le k}(2h(M_{i_1}\cap M_{i_2}) - 1) \\ &+ \cdots \\ &+ (-1)^{k-1}\left(2h\left(\bigcap_{i=1}^{k} M_i\right) - 1\right).\end{aligned}$$

The constant terms cancel by the identity

$$(1-1)^k = 0.$$

Hence,

$$h(G) = 2\left(\sum_{i=1}^{k} h(M_i) - \sum_{1\le i_1<i_2\le k} h(M_{i_1}\cap M_{i_2}) + \cdots + (-1)^{k-1} h\left(\bigcap_{i=1}^{k} M_i\right)\right). \quad (4.2)$$

Then,

$$h(D_{2n}) = \frac{2^{m_1}}{p_1 - 1}\left(p_1^{m_1+1} + p_1 - 2\right), \qquad \text{if } n = p_1^{m_1}.$$

In particular, $h(D_4) = 8, h(D_6) = 10$ *and* $h(D_8) = 32$

4.3 Enumeration For S_3 and S_4

Result 4.3.1. *The number* $g(S_3)$ *of all maximal chains of subgroups of the symmetric group* S_3 *is* 4 *and The number* $h(S_3)$ *of the symmetric group* S_3 *is* 10

The case $n = 3$ *is trivial since the symmetric group* S_3 *is isomorphic to the dihedral group* D_6. *Hence,*

$$g(S_3) = g(D_6) = 4$$

and

$$h(S_3) = h(D_6) = 10.$$

Theorem 4.3.1. *The number* $g(S_4)$ *of all maximal chains of subgroups of the symmetric group* S_4 *is* 44

Proof. To determine $g(S_4)$, we use the recurrence relation Equation 4.1

$$g(G) = \sum_{i=1}^{k} g(M_i),$$

where $M_1, M_2, \ldots, M_k$ are the maximal subgroups of G. Since $|S_4| = 24 = 2^3 \cdot 3$, the index of every maximal subgroup is a prime power.

Case 1. $[\mathbf{S_4} : \mathbf{M}] = \mathbf{2}$

Scanned with OKEN Scanner

In this case,there is a unique maximal subgroup of index 2, which is the alternating group A_4.Hence,

$$M \cong A_4.$$

Case 2. $[\mathbf{S_4 : M}] = \mathbf{2^2}$

Here $|M| = 6$. There are four such maximal subgroups, each stabilizing one element of $\{1, 2, 3, 4\}$. Hence, each is isomorphic to S_3.

Case 3. $[\mathbf{S_4 : M}] = \mathbf{2^3}$

In this case $|M| = 3$. Any subgroup of order 3 is contained in A_4. Therefore, such a subgroup cannot be maximal in S_4. Hence, there are no maximal subgroups of index 8.

Case 4. $[\mathbf{S_4 : M}] = \mathbf{3}$

Then $|M| = 8$. Such subgroups are Sylow 2-subgroups of S_4. There are three Sylow 2-subgroups, each isomorphic to the dihedral group D_8.

There are 3 such subgroups, they are ;

$$\begin{aligned} M_5 &= \langle (1234), (12)(34) \rangle, \\ M_6 &= \langle (1324), (13)(24) \rangle, \\ M_7 &= \langle (1342), (13)(24) \rangle. \end{aligned}$$

Then By Equation 4.1,

$$g(S_4) = g(A_4) + 4g(S_3) + 3g(D_8).$$

Hence,

$$g(S_4) = 44.$$

□

Theorem 4.3.2. *The number $h(S_4)$ of all distinct fuzzy subgroups of the symmetric group S_4 is* 232.

Proof. From the recurrence relation

$$h(S_4) = 2 \sum_{r=1}^{8} (-1)^{r-1} \sum_{0 \le i_1 < i_2 < \cdots < i_r \le 7} h(M_{i_1} \cap M_{i_2} \cap \cdots \cap M_{i_r})$$

Since the maximal subgroups of S_4 have already been identified, so now we list their intersections as follows,

$$\begin{aligned} M_0 \cap M_1 &= \langle (234) \rangle \cong \mathbb{Z}_3, \\ M_0 \cap M_2 &= \langle (134) \rangle \cong \mathbb{Z}_3, \\ M_0 \cap M_3 &= \langle (124) \rangle \cong \mathbb{Z}_3, \\ M_0 \cap M_4 &= \langle (123) \rangle \cong \mathbb{Z}_3. \end{aligned}$$

Scanned with OKEN Scanner

$$
\begin{aligned}
M_0 \cap M_5 &= M_0 \cap M_6 = M_0 \cap M_7 = M_5 \cap M_6 = M_5 \cap M_7 = M_6 \cap M_7 \\
&= M_0 \cap M_5 \cap M_6 = M_0 \cap M_5 \cap M_7 = M_0 \cap M_6 \cap M_7 \\
&= M_5 \cap M_6 \cap M_7 = M_0 \cap M_5 \cap M_6 \cap M_7 \\
&= \{e, (12)(34), (13)(24), (14)(23)\} \cong D_4,
\end{aligned}
$$

$$M_3 \cap M_4 = M_3 \cap M_6 = M_4 \cap M_6 = M_3 \cap M_4 \cap M_6 = \langle (12) \rangle \cong \mathbb{Z}_2,$$

$$M_2 \cap M_4 = M_2 \cap M_5 = M_4 \cap M_5 = M_2 \cap M_4 \cap M_5 = \langle (13) \rangle \cong \mathbb{Z}_2,$$

$$M_2 \cap M_3 = M_2 \cap M_7 = M_3 \cap M_7 = M_2 \cap M_3 \cap M_7 = \langle (14) \rangle \cong \mathbb{Z}_2,$$

$$M_1 \cap M_4 = M_1 \cap M_7 = M_4 \cap M_7 = M_1 \cap M_4 \cap M_7 = \langle (23) \rangle \cong \mathbb{Z}_2,$$

$$M_1 \cap M_3 = M_1 \cap M_5 = M_3 \cap M_5 = M_1 \cap M_3 \cap M_5 = \langle (24) \rangle \cong \mathbb{Z}_2,$$

$$M_1 \cap M_2 = M_1 \cap M_6 = M_2 \cap M_6 = M_1 \cap M_2 \cap M_6 = \langle (34) \rangle \cong \mathbb{Z}_2.$$

Now we focus on the terms appearing in the recurrence relation for $h(S_4)$. Define,

$$c_r = (-1)^{r-1} \sum_{0 \le i_1 < i_2 < \cdots < i_r \le 7} h(M_{i_1} \cap M_{i_2} \cap \cdots \cap M_{i_r}), \qquad r = 1, 2, \ldots, 8.$$

From the intersections we get,

$$
\begin{aligned}
c_8 &= -\binom{8}{8} = -1, \\
c_7 &= \binom{8}{7} = 8, \\
c_6 &= -\binom{8}{6} = -28, \\
c_5 &= \binom{8}{5} = 56, \\
c_4 &= -\left(\binom{8}{4} - 1 + h(D_4)\right) = -77, \\
c_3 &= \binom{8}{3} - 10 + 4h(D_4) + 6h(\mathbb{Z}_2) = 90, \\
c_2 &= -\left(6h(D_4) + 4h(\mathbb{Z}_3) + 18h(\mathbb{Z}_2)\right) = -92, \\
c_1 &= h(A_4) + 4h(S_3) + 3h(D_8) = 160.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
h(S_4) &= 2\sum_{r=1}^{8} c_r \\
&= 2(160 - 92 + 90 - 77 + 56 - 28 + 8 - 1) \\
&= 2(116) \\
&= 232.
\end{aligned}
$$

Hence, the number of distinct fuzzy subgroups of the symmetric group S_4 is

$$h(S_4) = 232.$$

□

Scanned with OKEN Scanner

4.4 Enumeration For S_n

For $n \geq 5$, the computation of the number of distinct fuzzy subgroups of the symmetric group S_n becomes considerably more difficult. This is because the subgroup structure of S_n is much more complicated than that of S_3 and S_4. But, some general observations can be made regarding the maximal subgroups of S_n.

Theorem 4.4.1. *For $n \geq 5$, the number $g(S_n)$ of all maximal chains of subgroups of the symmetric group S_n satisfies the inequality*

$$g(S_n) \geq 11(n-3)!\binom{n}{n-3} + \sum_{r=0}^{n-5} r!\binom{n}{r}\, g(A_{n-r}).$$

Proof. The total number of maximal chains of subgroups of S_n or chains of subgroups S_n ended in S_n is greater than the number of such chains contained in the union of all subgroup lattices $L(M_i)$, $i = 0, 1, ..., n$. We get,

$$g(S_n) \geq \sum_{i=0}^{n} g(M_i). \tag{4.3}$$

$$h(S_n) \geq 2\left(\sum_{i=0}^{n} h(M_i) - \sum_{0 \leq i_1 < i_2 \leq n} h(M_{i_1} \cap M_{i_2}) + \cdots + (-1)^n h\left(\bigcap_{i=0}^{n} M_i\right)\right). \tag{4.4}$$

Now, The equation 4.3 is equivalent to ,

$$g(S_n) \geq g(A_n) + n\, g(S_{n-1}).$$

Multiplying each of them by

$$(n-5)!\binom{n}{n-5},\ (n-6)!\binom{n}{n-6},\ \ldots$$

Respectively, and summing up all these inequalities, we get the result,

$$g(S_n) \geq 11(n-3)!\binom{n}{n-3} + \sum_{r=0}^{n-5} r!\binom{n}{r}\, g(A_{n-r}).$$

In particular,

$$g(S_n) > 11(n-3)!\binom{n}{n-3} = \frac{11}{6}n!.$$

□

Theorem 4.4.2. *For $n \geq 5$, the number $h(S_n)$ of all distinct fuzzy subgroups of the symmetric group S_n satisfies the following inequality:*

$$h(S_n) \geq 2\left[\sum_{r=0}^{n}(-1)^r\binom{n}{r}h(A_{n-r}) + \sum_{r=0}^{n-1}(-1)^r\binom{n}{r+1}h(S_{n-r-1})\right]$$

Scanned with OKEN Scanner

Proof. For $n \geq 5$, we consider the maximal subgroups of S_n, namely

$$M_0 = A_n, \quad M_i = \{\sigma \in S_n \mid \sigma(i) = i\},\ i = 1, 2, \ldots, n,$$

where $M_i \cong S_{n-1}$ for $i \geq 1$. From Equation 4.4

$$M_{i_1} \cap \cdots \cap M_{i_r} \cong \begin{cases} A_{n-r+1}, & i_1 = 0, \\ S_{n-r}, & i_1 \geq 1. \end{cases}$$

We get,

$$c_r = (-1)^{r-1} \sum h(M_{i_1} \cap \cdots \cap M_{i_r})$$

as

$$c_r = (-1)^{r-1} \left(\binom{n}{r-1} h(A_{n-r+1}) + \binom{n}{r} h(S_{n-r}) \right).$$

Therefore,

$$h(S_n) \geq 2 \left[\sum_{r=0}^{n-1} (-1)^r \binom{n}{r} h(A_{n-r}) + \sum_{r=0}^{n-1} (-1)^r \binom{n}{r+1} h(S_{n-r-1}) \right].$$

Hence proved. □

Scanned with OKEN Scanner

Conclusion

This project presented the fundamental concepts of fuzzy set theory and fuzzy subgroup theory, beginning with the definition of fuzzy sets and extending to the study of fuzzy subgroups and their basic properties. The correspondence between fuzzy subgroups and chains of level subgroups was discussed, providing a bridge between fuzzy group theory and classical group theory. The enumeration of fuzzy subgroups of finite cyclic groups was also studied, showing that the problem of counting distinct fuzzy subgroups can be transformed into the problem of counting subgroup chains.

The project further explored the classification of fuzzy subgroups of finite symmetric groups. Explicit results were obtained for the symmetric groups S_3 and S_4, while for S_n ($n \geq 5$) lower bound inequalities were established. These results demonstrate that the structure of a group plays a crucial role in determining its fuzzy subgroup structure.

Scanned with OKEN Scanner

Bibliography

1. L.A. Zadeh, "Fuzzy Sets," *Information and Control*, vol. 8, pp. 338-353, 1965.

2. A. Rosenfeld, "Fuzzy Groups," *Journal of Mathematical Analysis and Applications*, vol. 35, pp. 512-517, 1971.

3. M. Tarnauceanu "On the Number of Fuzzy Subgroups of Finite Symmetric Groups," *Journal of Multiple-Valued Logic & Soft Computing*, vol. 21, pp. 201-213, 2013.

4. Raden Sulaiman and Abd Ghafur Ahmad, "Counting Fuzzy Subgroups of Symmetric Groups S_2, S_3 and Alternating Group A_4," *Journal of Quality Measurement and Analysis*, vol. 6, pp. 987-994, 2011.

5. W.B. Vasantha Kandasamy, *Smarandache Fuzzy Algebra*, American Research Press, pp. 9-31, 2003.

Scanned with OKEN Scanner