A STUDY ON NON-EXPANSIVE MAPPINGS

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A Study On Non-Expansive Mappings

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CONTENTS

INTRODUCTION

The concept of a *fixed point*, a point that remains unchanged under a given mapping, is one of the fundamental ideas in modern mathematics. Fixed point theory has become an indispensable tool in nonlinear functional analysis, topology, optimization, differential equations, game theory, and economics. For a mapping $T : X \to X$, where X is a nonempty set, a point $x \in X$ is called a *fixed point* of T if $T(x) = x$. The existence and approximation of such points form the central theme of fixed point theory.

Among the various classes of mappings, *nonexpansive mappings* occupy a prominent position because of their wide range of applications and rich mathematical structure. A mapping T defined on a metric or Banach space X is said to be *nonexpansive* if

$$\|T(x) - T(y)\| \leq \|x - y\|, \qquad \forall x, y \in X.$$

Unlike contraction mappings, nonexpansive mappings do not necessarily satisfy a strict contractive condition. Consequently, classical results such as the Banach Contraction Principle are not directly applicable, making the study of nonexpansive mappings more challenging and requiring additional geometric and topological assumptions on the underlying space.

An important geometric condition in the study of the convergence of iterative sequences is the *Opial condition*. Introduced by Z. Opial, this condition plays a significant role in establishing the weak convergence of sequences generated by iterative methods for nonexpansive mappings. In Banach spaces satisfying Opial's condition, weak limits of iterative sequences exhibit a uniqueness property, making it an essential tool in convergence analysis.

Another fundamental result is the *Demiclosedness Principle*, which states that under suitable conditions, the operator $I - T$, where T is a nonexpansive mapping, is demiclosed at zero. This principle provides an important connection between weak convergence and fixed point theory by ensuring that if a sequence converges weakly and its residuals converge strongly to zero, then the weak limit is a fixed point of the mapping. It serves as a key ingredient in proving many convergence theorems for iterative algorithms.

This project presents a systematic study of fixed point theorems for nonexpansive mappings, beginning with the fundamental concepts of metric and Banach spaces, convexity, and nonexpansive mappings. It explores classical fixed point results, the role of geometric conditions such as normal structure and uniform convexity, the significance of Opial's condition and the Demiclosedness Principle, and their applications in convergence theory.

Chapter 1

PRELIMINARIES

1.1 Basic Concepts and Definitions

1.1.1 Metric Spaces

A *metric space* is a set X together with a function

$$d : X \times X \to \mathbb{R}$$

called a *metric*, which satisfies the following properties for all $x, y, z \in X$:

1. $d(x, y) \geq 0$ (non-negativity)
2. $d(x, y) = 0 \iff x = y$ (identity of indiscernibles)
3. $d(x, y) = d(y, x)$ (symmetry)
4. $d(x, z) \leq d(x, y) + d(y, z)$ (triangle inequality)

1.1.2 Banach Spaces

A *Banach space* is a normed linear space $(X, \|\cdot\|)$ that is complete with respect to the metric induced by the norm. That is, every Cauchy sequence $\{x_n\}$ in X converges to an element $x \in X$, i.e.,

$$\lim_{n\to\infty} \|x_n - x\| = 0.$$

1.1.3 Lipschitzian, Contraction and Non-expansive Maps

Let (X, d) be a metric space. A mapping $F : X \mapsto X$ is said to be *Lipschitzian* if there exists a constant $\alpha \geq 0$ such that

$$d(F(x), F(y)) \leq \alpha\, d(x, y), \qquad \forall\, x, y \in X.$$

A Lipschitzian mapping is necessarily continuous. The smallest value of α for which the above inequality holds is called the *Lipschitz constant* of F and is usually denoted by L.

- If $L < 1$, then F is called a *contraction mapping.*
- If $L = 1$, then F is called a *non-expansive mapping.*

Definition 1.1.1 *Let (X, d) be a metric space and let (x_n) be a sequence in X. The sequence (x_n) is said to be a* ***Cauchy sequence*** *if for every $\varepsilon > 0$ there exists a positive integer N such that $d(x_n, x_m) < \varepsilon$ for all $n, m \geq N$.*

Definition 1.1.2 *A metric space (X, d) is said to be* ***complete*** *if every Cauchy sequence in it converges to an element of it.*

Definition 1.1.3 *Let $X = (X, d)$ and $Y = (Y, d)$ be metric spaces. A mapping $f : X \to Y$ is said to be* ***continuous*** *at a point $x_0 \in X$ if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $d(x, x_0) < \delta$ impliesd$(f(x), f(x_0)) < \varepsilon$ for all x satisfying $d(x, x_0) < \delta$.The mapping f is said to be* ***continuous*** *if it is continuous at every point of X.*

Definition 1.1.4 *Let $X = (X, d)$ and $Y = (Y, d)$ be metric spaces. A mappingf $:$ $X \to Y$ is said to be* ***uniformly continuous*** *if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $d(f(x), f(y)) < \varepsilon$ whenever $d(x, y) < \delta$*

Definition 1.1.5 *Let $a \in X$, where (X, d) is a metric space. Then for each $r > 0$, $U(a, r) = \{x \in X : d(a, x) < r\}$ is a neighborhood or an open ball of center a and radius r.*

Definition 1.1.6 *Let (X, d) be a metric space. Then $G \subset X$ is called **open** if and only if every point of G has a neighborhood contained in G. Symbolically, we require that for every $x \in G$, there exists $r > 0$ such that $U(x, r) \subset G$.*

Definition 1.1.7 *A **topological space** is an ordered pair (X, τ) where τ is a collection of subsets of X satisfying the following properties:*

1. *The empty set and X belong to τ.*
2. *Any arbitrary (finite or infinite) union of members of τ belongs to τ.*
3. *The intersection of any finite number of members of τ belongs to τ.*

*The elements of τ are called **open sets** and the collection τ is called a **topology** on X.*

1.2 Banach Contraction Principle

Theorem 1.2.1 (Banach Contraction Principle) *Let (X, d) be a complete metric space and let $F : X \to X$ be a contraction with Lipschitz constant L. Then F has a unique fixed point $u \in X$. Furthermore, for any $x \in X$, we have*

$$\lim_{n\to\infty} F^n(x) = u$$

with

$$d(F^n(x), u) \leq \frac{L^n}{1-L} d(x, F(x)).$$

Proof: We first show uniqueness. Suppose there exist $x, y \in X$ with$x = F(x)$ and $y = F(y)$. Then,

$$d(x, y) = d(F(x), F(y)) \leq Ld(x, y).$$

Since $0 < L < 1$, this implies $d(x, y) = 0 \Rightarrow x = y$.
To show existence, select $x \in X$.
We first show that $\{F^n(x)\}$ is a Cauchy sequence. Notice for $n \in \{0, 1, \ldots\}$,

$$d(F^{n+1}(x), F^n(x)) \leq Ld(F^n(x), F^{n-1}(x)) \leq \cdots \leq L^n d(x, F(x)).$$

Thus for $m > n$, where $n \in \{0, 1, \ldots\}$,

$$\begin{aligned} d(F^m(x), F^n(x)) &\le d(F^m(x), F^{m-1}(x)) + \cdots + d(F^{n+1}(x), F^n(x)) \\ &\le L^{m-1} d(x, F(x)) + \cdots + L^n d(x, F(x)) \\ &= (L^n + L^{n+1} + \cdots + L^{m-1}) d(x, F(x)) \\ &\le \frac{L^n}{1-L}\, d(x, F(x)). \end{aligned}$$

That is, for $m > n$,

$$d(F^m(x), F^n(x)) \le \frac{L^n}{1-L}\, d(x, F(x)).$$

This shows that $\{F^n(x)\}$ is a Cauchy sequence, and since X is complete, there exists $u \in X$ with

$$\lim_{n\to\infty} F^n(x) = u.$$

Moreover the continuity of F yields,

$$u = \lim_{n\to\infty} F^{n+1}(x) = \lim_{n\to\infty} F(F^n(x)) = F(u).$$

Therefore, u is a fixed point of F. Finally, letting $m \to \infty$, we get

$$d(F^n(x), u) \le \frac{L^n}{1-L}\, d(x, F(x)).$$

1.2.1 Example for Banach's Contraction Principle

Example: Let $X = \mathbb{R}$, and define the mapping $T : X \to X$ by

$$T(x) = \frac{1}{2}x + 1.$$

We want to apply **Banach's Contraction Principle** to this mapping.

Step 1: Show that T is a contraction

Let $x, y \in \mathbb{R}$. Then,

$$|T(x) - T(y)| = \left| \frac{1}{2}x + 1 - \left(\frac{1}{2}y + 1 \right) \right| = \left| \frac{1}{2}(x - y) \right| = \frac{1}{2}|x - y|.$$

So, T is a contraction mapping with contraction constant $k = \frac{1}{2} < 1$.

Step 2: Use Banach's Contraction Principle

Since

- $\mathbb{R}$ is a complete metric space,
- and T is a contraction,

by **Banach's Fixed Point Theorem**, T has a unique fixed point in $\mathbb{R}$.

Step 3: Find the fixed point

We solve

$$T(x) = x \Rightarrow \frac{1}{2}x + 1 = x \Rightarrow 1 = x - \frac{1}{2}x = \frac{1}{2}x \Rightarrow x = 2.$$

Conclusion: The unique fixed point of the mapping $T(x)$ is $x = 2$

1.3 Brouwer Fixed Point Theorem

Theorem 1.3.1 (Brouwer) *Let $f : D^n \to D^n$ be a continuous function from the closed n-dimensional disk to itself. Then f has at least one fixed point. That is, there exists $x \in D^n$ such that $f(x) = x$, where*

$$D^n = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$$

is the closed unit ball in $\mathbb{R}^n$.

Proof: Assume, for the sake of contradiction, that the continuous function

$$f : D^2 \to D^2$$

has no fixed point, i.e.,$f(x) \neq x$ for all $x \in D^2$.

We will now construct a retraction from D^2 onto its boundary S^1, which contradicts a known topological result that no such retraction exists.

Since $f(x) \neq x$ for all $x \in D^2$, for each $x \in D^2$ we can draw a unique ray starting at $f(x)$ and passing through x. This ray intersects the boundary circle S^1 of the disk D^2 at exactly one point.

Let us define a map

$$r : D^2 \to S^1$$

such that $r(x)$ is the point where this ray intersects the boundary S^1. This map r is continuous, and it agrees with the identity map on the boundary S^1, i.e., $r(x) = x$ for all $x \in S^1$. Hence, r is a retraction from D^2 onto S^1.

But it is a well-known fact from algebraic topology that there is no continuous retraction from D^2 onto S^1. Therefore, we arrive at a contradiction.

Hence, our assumption that f has no fixed point must be false. Thus, there exists at least one point $x \in D^2$ such that

$$f(x) = x.$$

Example 1.3.1 (Unit Disk) *Let D be the unit disk in $\mathbb{R}^2$,*

$$D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}.$$

Define the function

$$f(x, y) = \left(\frac{x}{2}, \frac{y}{2}\right).$$

This function ***shrinks every point*** *toward the origin $(0, 0)$. It is continuous and maps the disk into itself, since for any $(x, y) \in D$,*

$$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{2}\right)^2 = \frac{x^2 + y^2}{4} \leq \frac{1}{4} < 1.$$

Fixed Point: *The origin $(0, 0)$ is a fixed point because*

$$f(0, 0) = \left(\frac{0}{2}, \frac{0}{2}\right) = (0, 0).$$

[**Real-world Example (Stirring Coffee)**] Imagine stirring a cup of coffee.

The coffee is inside a circular cup, which can be modeled as a two-dimensional disk. When you stir it, every point of the liquid moves according to a continuous function.

Brouwer's Fixed Point Theorem guarantees that at least one point of the liquid remains unmoved—this is known as a *fixed point.*

Fixed Point: Some unmoved droplet of coffee.

Chapter 2

NON-EXPANSIVE MAPPING

2.1 Non-expansive maps

Mappings that are defined in metric spaces and do not increase the distances between pairs of points and their images are called nonexpansive.

Definition 2.1.1 *If X is a Banach space and $D \subseteq X$,then a mapping $T : D \mapsto X$ is said to be nonexpansive if for each $x, y \in D$*

$$\|T(x) - T(y)\| \leq \|x - y\|$$

Example 2.1.1 *Define the mapping*

$$T : \mathbb{R} \to \mathbb{R}, \qquad T(x) = \frac{x}{2}.$$

For any $x, y \in \mathbb{R}$,

$$\|T(x) - T(y)\| = \left\|\frac{x}{2} - \frac{y}{2}\right\| = \frac{1}{2}\|x - y\| \leq \|x - y\|.$$

Hence,

$$\|T(x) - T(y)\| \leq \|x - y\|.$$

*which shows that $T(x) = \frac{x}{2}$ is a **non-expansive mapping**.*

Theorem 2.1.1 *If K is a bounded closed and convex subset of a uniformly convex Banach space X and if $T : K \mapsto K$ is nonexpansive (ie,$\|T(x) - T(y)\| \leq \|x - y\|$ for each $x, y \in K$),then T has a fixed point.*

The existence of fixed points for nonexpansive mappings has generally fallen into three categories.

We say that a Banach Space has the Fixed point property(FPP) if each of its nonempty,bounded,closed,and convex subsets has the fixed point property for nonexpansive self-mappings.

It is said to have the Weakly Compact Fixed point property(Wc-FPP) if each of its weakly compact covex subsets has the fixed point property.

Finally,a Banach space has the Ball fixed point property(B-FPP) if its closed unit ball has the fixed point property.

2.2 Subclasses of Nonexpansive Mappings

Various subclasses of nonexpansive mappings arise in natural ways. when $D \subset X$ is a convex,closed and bounded set,the family F of nonexpansive mapping $T : D \mapsto D$ can be viewed as a convex,closed,and bounded subset of the space $C[D, X]$, of all continous mappings from D into X,furnished with the natural uniform norm:

$$\|T\|_{C[D,X]} = \sup \{\|Tx\| : x \in D\}.$$

The Subfamily $F_0 \subset F$ of all contractions is dense in F in the topology generated by this norm.

From the Banach contraction Principle, D has the fixed point property with respect to the whole family F. There are several other subclasses of F.

Theorem 2.2.1 *Let D be a weakly compact convex subset convex subset of a super reflexive Banach space .Then D has the fixed point property relative to the class of isometries.*

2.3 Various Subclasses

Isometries: The isometries are mappings $T : D \to D$ satisfying

$$\|Tx - Ty\| = \|x - y\| \qquad \forall x, y \in D.$$

Contractive mappings: These are mappings satisfying

$$\|Tx - Ty\| < \|x - y\| \qquad \forall x, y \in D \text{ with } x \neq y.$$

This condition implies two things:

- A contractive mapping T has at most one fixed point.
- If T is fixed point free and D is weakly compact, then T has exactly one minimal closed convex invariant set.

Remark: D has the fixed point property for contractive mappings if D is weakly compact and such that for any closed convex $K \subset D$, there exists a point $z \in D$ (not necessarily $z \in K$) such that

$$\sup\{\|z - x\| : x \in K\} < \operatorname{diam}(K).$$

This condition is a modification of normal structure

.

Mappings of Type τ: Let τ denote the set of strictly increasing convex functions $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\varphi(0) = 0$. A mapping $T : D \to D$ is said to be of type τ if there exists $\varphi \in \tau$ such that

$$\varphi\Big(\big\|\varepsilon Tx + (1-\varepsilon)Ty - T\big(\varepsilon x + (1-\varepsilon)y\big)\big\|\Big) \leq \|x - y\| - \|Tx - Ty\|,$$

$$\forall x, y \in D, \quad \varepsilon \in (0,1).$$

- Mappings of type τ are nonexpansive.
- Affine nonexpansive mappings are of type τ.
- The fixed point set of a mapping of type τ is convex.

Firmly Nonexpansive Mappings: A mapping $T : D \to D$ is said to be firmly nonexpansive if for all $x, y \in D$, the function

$$\phi_{x,y}(t) = \|(1-t)(x-y) + t(Tx - Ty)\|, \qquad t \in [0,1]$$

is nonincreasing on $[0,1]$.

Remark: Any firmly nonexpansive mapping is nonexpansive, but the converse is not true.

Example: If D is a subset of a Hilbert space H, then $T : D \to D$ is firmly nonexpansive if and only if it is of the form

$$T = \frac{1}{2}(I + G),$$

where $G : D \to H$ is nonexpansive.

Rotative Mappings: If $T : D \to D$ is nonexpansive, then for any $n \in \mathbb{N}$, we have

$$\|x - T^n x\| \leq n\|x - Tx\|.$$

A mapping T is said to be (n,α)-rotative with $\alpha < n$ if for all $x \in D$,

$$\|x - T^n x\| \leq \alpha\|x - Tx\|.$$

T is called *rotative* if it is (n,α)-rotative for some pair (n,α) with $\alpha < n$.

- All contractions are rotative.
- There are rotative isometries.
- There exist nonexpansive mappings which are not (n,α)-rotative for any (n,α).

Remark: In contrast to the general case, the fixed point property for rotative nonexpansive mappings does not depend on the geometry of the set D. Any closed convex set D, regardless of whether it is bounded or not, has this property.

2.4 Normal Structure

Let C be a nonempty bounded subset of a Banach space X. Then a point $x_0 \in C$ is said to be

(i) a *diametral point* of C if

$$\sup\{\|x_0 - x\| : x \in C\} = \operatorname{diam}(C),$$

(ii) a *nondiametral point* of C if

$$\sup\{\|x_0 - x\| : x \in C\} < \operatorname{diam}(C).$$

A nonempty convex subset C of a Banach space X is said to have *normal structure* if each convex bounded subset D of C with at least two points contains a nondiametral point, i.e., there exists $x_0 \in D$ such that

$$\sup\{\|x_0 - x\| : x \in D\} < \operatorname{diam}(D).$$

Geometrically, C is said to have normal structure if for each convex bounded subset D of C with $\operatorname{diam}(D) > 0$, there exist a point $x_0 \in D$ and $r < \operatorname{diam}(D)$ such that

$$D \subseteq B_r[x_0].$$

The Banach space X is said to have normal structure if every closed convex bounded subset C of X with $\operatorname{diam}(C) > 0$ has normal structure.

Theorem 2.4.1 *Every compact convex subset C of a Banach space X has normal structure.*

proof: Suppose, for contradiction, that C does not have normal structure. Let D be a convex subset of C that has at least two points. Because C does not have normal structure, all points of D are diametral. Now we construct a sequence $\{x_i\}_{i=1}^{\infty}$ in D such that

$$\|x_i - x_j\| = \operatorname{diam}(D) \quad \text{for all } i, j \in \mathbb{N},\ i \neq j.$$

For this, let x_1 be an arbitrary point in D. Then there exists a point $x_2 \in D$ such that

$$\operatorname{diam}(D) = \|x_1 - x_2\|.$$

Because D is convex, there exists a point

$$\frac{x_1 + x_2}{2} \in D.$$

Next we choose a point $x_3 \in D$ such that

$$\operatorname{diam}(D) = \left\| x_3 - \frac{x_1 + x_2}{2} \right\|.$$

Proceeding in the same manner, we obtain a sequence $\{x_n\}$ in D such that

$$\operatorname{diam}(D) = \left\| x_{n+1} - \frac{x_1 + x_2 + \cdots + x_n}{n} \right\|, \qquad n \geq 2.$$

Because

$$\begin{aligned}
\operatorname{diam}(D) &= \left\| x_{n+1} - \frac{x_1 + x_2 + \cdots + x_n}{n} \right\| \\
&= \left\| \frac{(x_{n+1} - x_1) + (x_{n+1} - x_2) + \cdots + (x_{n+1} - x_n)}{n} \right\| \\
&\leq \frac{1}{n} \Big(\|x_{n+1} - x_1\| + \|x_{n+1} - x_2\| + \cdots + \|x_{n+1} - x_n\| \Big) \\
&\leq \operatorname{diam}(D),
\end{aligned}$$

it follows that $\operatorname{diam}(D) = \|x_{n+1} - x_i\|, \qquad 1 \leq i \leq n$. This implies that the sequence $\{x_n\}$ has no convergent subsequences. This contradicts the compactness of C.

corollary 2.4.1 *Every finite-dimensional Banach space has normal structure.*

Theorem 2.4.2 *Every closed convex bounded subset C of a uniformly convex Banach space X has normal structure.*

proof: Let D be a closed convex subset of C with $\operatorname{diam}(D) = d > 0$. Let x_1 be an arbitrary point in D. Choose a point $x_2 \in D$ such that $\|x_1 - x_2\| \geq \frac{d}{2}$. Because D is convex,

$$\frac{x_1 + x_2}{2} \in D.$$

Set

$$x_0 = \frac{x_1 + x_2}{2}.$$

By the uniform convexity,

$$\|u\| \leq r,\ \|v\| \leq r \text{ and } \|u - v\| \geq \varepsilon > 0 \Rightarrow \left\|\frac{u+v}{2}\right\| \leq \left(1 - \delta_X\left(\frac{\varepsilon}{r}\right)\right) r.$$

Hence for $x \in D$ we have

$$\begin{aligned}
\|x - x_0\| &= \left\| x - \frac{x_1 + x_2}{2} \right\| \\
&= \left\| \frac{(x - x_1) + (x - x_2)}{2} \right\| \\
&\leq d\left(1 - \delta_X\left(\frac{d}{2d}\right)\right) \qquad (1.1) \\
&= d\left(1 - \delta_X\left(\frac{1}{2}\right)\right) \\
&< d, \qquad (\text{as } \delta_X(1/2) > 0).
\end{aligned}$$

Consequently,

$$\sup\{\|x - x_0\| : x \in D\} < \operatorname{diam}(D).$$

Example 2.4.1 *The space $C[0,1]$ of continuous real-valued functions with "sup" norm does not have normal structure.*

For that consider the subset C of $X = C[0,1]$ defined by

$$C = \{f \in C[0,1] : 0 = f(0) \leq f(t) \leq f(1) = 1,\ t \in [0,1]\}.$$

Let $f_1, f_2 \in C$ and $\lambda \in [0,1]$ and $f = \lambda f_1 + (1-\lambda) f_2$. Then $f(0) = 0,\ f(1) = 1$ and $0 \leq f(t) \leq 1$ for all $t \in [0,1]$. Hence C is convex. Thus, C is a closed convex bounded subset of X with $\operatorname{diam}(C) = \sup\{\|f - g\| : f, g \in C\} = 1$. Then each point of C is a diametral point. In fact, for $f_0 \in C$,

$$\sup\{\|f_0 - f\| : f \in C\} = 1 = \operatorname{diam}(C).$$

Therefore, C does not have normal structure.

A bounded sequence $\{x_n\}$ in a Banach space is said to be a **diametral sequence** if

$$\lim_{n\to\infty} d(x_{n+1}, \mathrm{co}\{x_1, x_2, \cdots, x_n\}) = \mathrm{diam}(\{x_n\}),$$

where

$$d(x, A) = \inf_{y\in A} d(x, y).$$

Remarks 2.4.1 *Any subsequence of a diametral sequence is also diametral*

Theorem 2.4.3 *Let X be a reflexive Banach space and suppose K is a bounded closed convex subset of X which has normal structure. Then any nonexpansive mapping $T : K \to K$ has a fixed point. In particular, if X has normal structure, then X has the Fixed Point Property (FPP).*

Proof: Let

$$\mathcal{V} = \{D \subset K : D \text{ is closed, convex, } D \neq \emptyset,\ T(D) \subset D\}.$$

Since the members of $\mathcal{V}$ are weakly compact, each descending chain in $\mathcal{V}$ has a lower bound (namely, the intersection of its members). Hence, by Zorn's Lemma, $\mathcal{V}$ has a minimal element, say D_0.

Now,

$$T(D_0) \subset D_0 \Rightarrow T\big(\mathrm{co}(T(D_0))\big) \subset \mathrm{co}(T(D_0)).$$

Thus, $\mathrm{co}(T(D_0)) \in \mathcal{V}$, and by the minimality of D_0, we must have $\mathrm{co}(T(D_0)) = D_0$.

Let $u \in C(D_0)$, the Chebyshev center of D_0; then

$$r_u(D_0) = r(D_0).$$

Since T is nonexpansive, we have

$$\|T(u) - T(v)\| \leq \|u - v\| \leq r(D_0), \qquad \forall v \in D_0,$$

which implies

$$T(D_0) \subset B(T(u), r(D_0)).$$

Therefore,

$$D_0 = \mathrm{co}(T(D_0)) \subset B(T(u), r(D_0)),$$

which gives

$$r_{T(u)}(D_0) = r(D_0),$$

so $T(u) \in C(D_0)$. By the minimality of D_0, this implies $C(D_0) = D_0$.

Thus,

$$\operatorname{diam}(D_0) \leq r(D_0),$$

and since K has normal structure, it must be the case that D_0 consists of a single point, which is fixed under T.

2.5 Asymptotic Normal Structure

It has been known for some time that even in reflexive Banach spaces, normal structure is not essential for the Fixed Point Property (FPP). An example is provided by the spaces X_β for $\beta > 0$, defined by

$$X_\beta = \left\{ x \in \ell^2 : \|x\|_\beta = \max\{\|x\|_2, \beta\|x\|_\infty\} < \infty \right\}.$$

Some facts about X_β:

(i) X_β is reflexive (since it is isomorphic to the Hilbert space ℓ^2).

(ii) X_β has normal structure if and only if $\beta < \sqrt{2}$.

(iii) X_β has asymptotic normal structure if and only if $\beta < 2$.

The concept of *asymptotic normal structure* was introduced by Baillon and Schoneberg in 1981. A Banach space X is said to have asymptotic normal structure if each nonempty bounded closed convex subset K of X containing more than one point satisfies the following:

If $\{x_n\} \subset K$ is such that

$$\|x_n - x_{n+1}\| \to 0,$$

then there exists $x \in K$ such that

$$\liminf_{n\to\infty} \|x_n - x\| < \operatorname{diam}(K).$$

Note:

In 1971, it was observed by Day, James, and Swaminathan that every separable Banach space has an equivalent norm which has normal structure. Thus, every separable reflexive space admits an equivalent norm under which it has the Fixed Point Property (FPP).

The question of whether reflexivity is essential for FPP remains open, but there is compelling evidence suggesting that it might be. For example, it is known that certain bounded closed convex subsets of classical nonreflexive spaces like c_0 and ℓ^1 fail to have the Fixed Point Property.

Moreover, Bessaga and Pełczyński have shown that if X is a Banach space with an unconditional basis, then X is reflexive if and only if X contains a subspace isomorphic to c_0 or ℓ^1.

Hence, all classical nonreflexive spaces can be renormed in such a way that they fail to have the Fixed Point Property.

The Space L^1. As we have noted, ℓ^1 (hence L^1) fails to have the Fixed Point Property (FPP). However, in 1981, Alspach proved much more, namely that L^1 fails to have the weakly compact Fixed Point Property (wc-FPP).

At the same time, Maurey proved that all reflexive subspaces of L^1 do have FPP (hence wc-FPP).

There has been another quite recent development. Dowling, Lennard, and Turett (1993) have shown that non-reflexive subspaces of L^1 fail to have FPP.

Thus, a subspace of L^1 has the Fixed Point Property if and only if it is reflexive.

2.6 The Goebel–Karlovitz Lemma

Lemma 2.6.1 *Each point of K is a diametral point; that is,*

$$\sup\{\|x-y\| : y \in K\} = \operatorname{diam}(K)$$

for each $x \in K$. Now let K be bounded, closed, and convex, fix $z \in K$, and let $T : K \to K$ be nonexpansive. Then for $t \in (0,1)$, the mapping $T_t : K \to K$ defined *by*

$$T_t(x) = (1-t)z + tT(x)$$

is a contraction mapping with a unique fixed point. Since

$$\lim_{t\to 1} \|T_t(x) - T(x)\| = \lim_{t\to 1} \|(1-t)z - (1-t)T(x)\| = 0,$$

we have the following:

Lemma 2.6.2 *If K is a bounded, closed, and convex subset of a Banach space, and if $T : K \to K$ is nonexpansive, then there exists a sequence $\{x_n\} \subset K$ such that*

$$\lim_{n\to\infty} \|x_n - T(x_n)\| = 0.$$

Lemma 2.6.3 (Goebel–Karlovitz) *Let K be a subset of a Banach space X, and suppose K is minimal invariant with respect to being nonempty, weakly compact, convex, and T-invariant for some nonexpansive mapping T.Suppose $\{x_n\} \subset K$ satisfies*

$$\lim_{n\to\infty} \|x_n - T(x_n)\| = 0.$$

Then for each $x \in K$,

$$\lim_{n\to\infty} \|x - x_n\| = \mathrm{diam}(K).$$

Proof: As we have just seen, if $\mathrm{diam}(K) > 0$, then K can have no diametral points.

Now suppose there exists a sequence $\{x_n\} \subset K$ for which

$$\lim_{n\to\infty} \|x_n - T(x_n)\| = 0,$$

but for which

$$\lim_{n\to\infty} \|x - x_n\| = r < \mathrm{diam}(K)$$

for some $x \in K$. Let

$$C = \left\{ z \in K : \limsup_{n\to\infty} \|z - x_n\| \le r \right\}.$$

It is easy to see that C is convex. Also, C is closed. Indeed, if $\{u_i\} \subset C$ and

$$\lim_{i\to\infty} u_i = u,$$

then for each $i = 1, 2, \ldots,$

$$\limsup_{n\to\infty} \|u - x_n\| \leq \limsup_{n\to\infty} \|u - u_i\| + \limsup_{n\to\infty} \|u_i - x_n\| \leq \|u - u_i\| + r.$$

Letting $i \to \infty$, we obtain

$$\limsup_{n\to\infty} \|u - x_n\| \leq r.$$

Thus $u \in C$, and so C is closed. Also, if $u \in K$,

$$\limsup_{n\to\infty} \|T(u) - x_n\| \leq \limsup_{n\to\infty} \big(\|T(u) - T(x_n)\| + \|T(x_n) - x_n\|\big) \leq \limsup_{n\to\infty} \|u - x_n\| \leq r.$$

This proves that

$$T(C) \subset C.$$

Since any closed convex subset of a weakly compact set is itself weakly compact, and since K is minimal with respect to being weakly compact, convex, and T-invariant, we conclude that

$$C = K.$$

However, if $\operatorname{diam}(K) > 0$ and if $\varepsilon > 0$ is chosen so that

$$r + \varepsilon < \operatorname{diam}(K),$$

then the family

$$\{B(x; r + \varepsilon) : x \in K\}$$

has the finite intersection property. Since these balls are weakly compact, we have

$$\bigcap_{x \in K} B(x; r + \varepsilon) \neq \emptyset.$$

But this implies

$$\operatorname{diam}(K) \leq r + \varepsilon < \operatorname{diam}(K),$$

a contradiction. Thus the lemma is proved.

Chapter 3

THE OPIAL CONDITION AND DEMICLOSEDNESS PRINCIPLE

The Opial condition plays an important role in convergence of sequences and in the study of the demiclosedness principle of nonlinear mappings and the geometry of Banach spaces.

A Banach space X is said to satisfy the *Opial condition* if whenever a sequence $\{x_n\}$ in X converges weakly to $x_0 \in X$, then

$$\liminf_{n\to\infty} \|x_n - x_0\| < \liminf_{n\to\infty} \|x_n - x\| \quad \text{for all } x \in X,\ x \neq x_0. \tag{3.1}$$

We observe that (3.1) is equivalent to the analogous condition obtained by replacing lim inf by lim sup. Replacing the strict inequality "<" in (3.1) with "≤", we obtain the definition of the so-called non-strict (or weak) Opial condition.

Example 3.0.1 *Every Hilbert space satisfies the Opial condition, i.e., if the sequence $\{x_n\}$ in a Hilbert space H converges weakly to $x \in H$, then*

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\| \qquad \textit{for all } y \in H,\ y \neq x.$$

In fact, because every weakly convergent sequence is necessarily bounded, we have that

$$\limsup_{n\to\infty} \|x_n - x\| \quad \textit{and} \quad \limsup_{n\to\infty} \|x_n - y\|$$

are finite. Note

$$\|x_n - y\|^2 = \|x_n - x + x - y\|^2 = \|x_n - x\|^2 + \|x - y\|^2 + 2\langle x_n - x,\ x - y\rangle,$$

so that

$$\limsup_{n\to\infty} \|x_n - y\|^2 > \limsup_{n\to\infty} \|x_n - x\|^2.$$

Example 3.0.2 *Let f be a periodic function with period 2π defined by*

$$f(t) = \begin{cases} 1, & \text{if } 0 \le t \le \frac{4\pi}{3}, \\ \\ -2, & \text{if } \frac{4\pi}{3} < t \le 2\pi. \end{cases}$$

Set

$$x_n(t) = f(nt), \qquad n \in \mathbb{N}.$$

Then $\{x_n\}$ is a weakly null sequence of Rademacher-like functions in $L_p[0, 2\pi]$ for every $1 < p < \infty$. Define a function

$$\lambda_p(s) = \lim_{n\to\infty} \|x_n - s\|^p = \int_0^{2\pi} |f(t) - s|^p \, dt,$$

where $s \in \mathbb{R}$ is treated as the constant function. Note

$$\lambda_p'(0) = -p \int_0^{2\pi} |f(t)|^{p-1} \operatorname{sgn}(f(t)) \, dt = \frac{4\pi}{3} \, p \left(2^{p-2} - 1\right),$$

$$\lambda_p'(0) \neq 0 \qquad \text{whenever } p \neq 2.$$

It follows that $\lambda_p(0)$ is not the minimal value of λ_p, except for the case $p = 2$. Therefore, $L_p[0, 2\pi]$ does not satisfy even the nonstrict Opial condition for any $p \neq 2$.

We now give a necessary and sufficient condition for a space satisfying the Opial condition.

Proposition 3.0.1 *A Banach space X satisfies the Opial condition if and only if*

$$x_n \rightharpoonup 0 \text{ and } \liminf_{n\to\infty} \|x_n\| = 1 \Rightarrow \liminf_{n\to\infty} \|x_n - x\| > 1 \quad \text{for all } x \neq 0. \tag{3.2}$$

Proof: Suppose that the condition (3.2) is satisfied.

Let

$$u_n \rightharpoonup u$$

and

$$r = \liminf_{n\to\infty} \|u_n - u\|.$$

If $r = 0$, then the Opial condition (3.1) follows from the uniqueness of a weak limit. If

$$r = \liminf_{n\to\infty} \|u_n - u\| > 0,$$

then

$$x_n = r^{-1}(u_n - u) \rightharpoonup 0$$

and

$$\liminf_{n\to\infty} \|x_n\| = 1.$$

Hence from (3.2) we have

$$\liminf_{n\to\infty} \|x_n - x\| > 1 \qquad \text{for } x \neq 0,$$

which implies that

$$\liminf_{n\to\infty} \|r^{-1}(u_n - u) - x\| > \liminf_{n\to\infty} \|r^{-1}(u_n - u)\|.$$

That is,

$$\liminf_{n\to\infty} \|u_n - (u + rx)\| > \liminf_{n\to\infty} \|u_n - u\|$$

for

$$u \neq u + rx.$$

Hence X satisfies the Opial condition. The inverse implication is obvious.

Proposition 3.0.2 *Let $X_1, X_2, \ldots, X_k$ be Banach spaces with norms $\|\cdot\|_1, \|\cdot\|_2, \ldots, \|\cdot\|_k$, respectively. Let p be a constant in $[1, \infty)$ and put*

$$X = X_1 \times X_2 \times \cdots \times X_k,$$

where the norm of X is given by

$$\|(x_1, x_2, \ldots, x_k)\| = (\|x_1\|_1^p + \|x_2\|_2^p + \cdots + \|x_k\|_k^p)^{1/p}$$

for all $(x_1, x_2, \ldots, x_k) \in X$.

Then the following are equivalent:

(a) X has the Opial condition.

(b) Each X_j has the Opial condition.

Proof: $(a) \Rightarrow (b)$.

Let $\{x_n\}$ be a sequence in X_j such that $x_n \rightharpoonup z$. Then the sequence

$$\{(0, 0, \ldots, 0, x_n, 0, \ldots, 0)\}$$

(with x_n in the jth position) converges weakly in X to

$$(0, 0, \ldots, 0, z, 0, \ldots, 0).$$

Using this fact, one can easily see that (a) implies (b).

Conversely, let $\{x_n\}$ be a sequence in X such that $x_n \rightharpoonup z$ and let

$$w \in X \setminus \{z\}.$$

Set

$$x_n = (x_n^{(1)}, x_n^{(2)}, \ldots, x_n^{(k)}),$$

$$z = (z^{(1)}, z^{(2)}, \ldots, z^{(k)}),$$

and

$$w = (w^{(1)}, w^{(2)}, \ldots, w^{(k)}).$$

Because $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\liminf_{n\to\infty} \|x_n - w\|^p = \lim_{i\to\infty} \|x_{n_i} - w\|^p,$$

and that the limits

$$\lim_{i\to\infty} \|x_{n_i}^{(j)} - z^{(j)}\|_j$$

exist for all $j = 1, 2, \ldots, k$. Because each X_j satisfies the Opial condition,

$$\lim_{i\to\infty} \|x_{n_i}^{(j)} - z^{(j)}\|_j^p \leq \liminf_{n\to\infty} \|x_n^{(j)} - w^{(j)}\|_j^p,$$

for $j = 1, 2, \ldots, k$, and

$$\lim_{i\to\infty} \|x_{n_i}^{(\ell)} - z^{(\ell)}\|_\ell^p < \liminf_{n\to\infty} \|x_n^{(\ell)} - w^{(\ell)}\|_\ell^p$$

for some ℓ, $1 \le \ell \le k$, because $z \ne w$. Therefore,

$$\begin{aligned}
\liminf_{n\to\infty} \|x_n - z\|^p &\le \lim_{i\to\infty} \|x_{n_i} - z\|^p \\
&= \sum_{j=1}^{k} \lim_{i\to\infty} \|x_{n_i}^{(j)} - z^{(j)}\|_j^p \\
&< \sum_{j=1}^{k} \liminf_{n\to\infty} \|x_n^{(j)} - w^{(j)}\|_j^p \\
&\le \lim_{i\to\infty} \|x_{n_i} - w\|^p \\
&= \liminf_{n\to\infty} \|x_n - w\|^p.
\end{aligned}$$

Therefore,

$$\liminf_{n\to\infty} \|x_n - z\| < \liminf_{n\to\infty} \|x_n - w\|.$$

Hence X satisfies the Opial condition.

Definition 3.0.1 *A Banach space X is said to satisfy the* uniform Opial condition *if for each $t > 0$, there exists an $r > 0$ such that*

$$1 + r \le \liminf_{n\to\infty} \|x_n + x\|$$

for each $x \in X$ with $\|x\| \ge t$ and each sequence $\{x_n\}$ in X such that

$$x_n \rightharpoonup 0 \quad \textit{and} \quad \liminf_{n\to\infty} \|x_n\| \ge 1.$$

Definition 3.0.2 *A Banach space X is said to satisfy the* locally uniform Opial condition *if for any weakly null sequence $\{x_n\}$ in X with*

$$\liminf_{n\to\infty} \|x_n\| \ge 1$$

and any $t > 0$, there is an $r > 0$ such that

$$1 + r \le \liminf_{n\to\infty} \|x_n + x\|$$

for every $x \in X$ with $\|x\| \ge t$.

Note that

uniform Opial condition $\Longrightarrow$ locally uniform Opial condition $\Longrightarrow$ Opial condition.

Definition 3.0.3 *We now define the Opial modulus of X, denoted by r_X, as follows:*

$$r_X(t) = \inf \left\{ \liminf_{n\to\infty} \|x_n + x\| - 1 \right\},$$

where $t \geq 0$ and the infimum is taken over all $x \in X$ with $\|x\| \geq t$ and sequences $\{x_n\}$ in X such that

$$x_n \rightharpoonup 0 \qquad \text{and} \qquad \liminf_{n\to\infty} \|x_n\| \geq 1.$$

It is obvious that r_X is nondecreasing and that X satisfies the uniform Opial condition if and only if

$$r_X(t) > 0 \qquad \text{for all } t > 0.$$

Note that in the definition of locally uniform Opial condition, "$\liminf$" can be replaced with "$\limsup$". Let $\{x_n\}$ be a weakly null sequence in a Banach space X with

$$\liminf_{n\to\infty} \|x_n\| \geq 1.$$

We define the local Opial modulus of X as follows:

$$r_{X,\{x_n\}}(t) = \inf \left\{ \liminf_{n\to\infty} \|x_n + x\| - 1 : x \in X,\ \|x\| \geq t \right\}.$$

X has the locally uniform Opial condition if

$$r_{X,\{x_n\}}(t) > 0 \qquad \text{for all } t > 0,$$

and the uniform Opial condition if

$$r_X(t) = \inf \left\{ r_{X,\{x_n\}}(t) : x_n \rightharpoonup 0,\ \liminf_{n\to\infty} \|x_n\| \geq 1 \right\} > 0$$

for all $t > 0$.

Proposition 3.0.3 *A Banach space X satisfies the locally uniform Opial condition if and only if for any sequence $\{x_n\}$ in X that converges weakly to $x \in X$ and for any sequence $\{y_m\}$ in X,*

$$\limsup_{m\to\infty} \left(\limsup_{n\to\infty} \|x_n - y_m\| \right) \leq \limsup_{n\to\infty} \|x_n - x\|$$

implies $y_m \to x$.

Proof: Assume that X satisfies the locally uniform Opial condition. Let $\{x_n\}$ be a sequence in X with $x_n \rightharpoonup x \in X$ and $\{y_m\}$ a sequence in X such that

$$\limsup_{m\to\infty}\left(\limsup_{n\to\infty}\|x_n - y_m\|\right) \le \limsup_{n\to\infty}\|x_n - x\|. \tag{3.3}$$

Set

$$d := \limsup_{n\to\infty}\|x_n - x\|.$$

If $d = 0$, then $y_m \to x$. If $d > 0$, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$d = \lim_{i\to\infty}\|x_{n_i} - x\|.$$

Suppose, for contradiction, that $\{y_m\}$ does not converge to x in norm. Then there exist an $\varepsilon > 0$ and a subsequence $\{y_{m_j}\}$ of $\{y_m\}$ such that

$$\|y_{m_j} - x\| \ge \varepsilon \qquad \text{for all } j \in \mathbb{N}.$$

Set

$$z_i := \frac{x_{n_i} - x}{d}.$$

By the locally uniform Opial condition of X, we have an $r > 0$ such that

$$1 + r \le \liminf_{i\to\infty}\|z_i + z\|$$

for all $z \in X$ with

$$\|z\| \ge \frac{\varepsilon}{d}.$$

In particular, we have

$$\liminf_{i\to\infty}\|x_{n_i} - y_{m_j}\| \ge d(1+r) \qquad \text{for all } j \in \mathbb{N},$$

which gives that

$$\limsup_{m\to\infty}\left(\limsup_{n\to\infty}\|x_n - y_m\|\right) \ge d(1+r) > d = \limsup_{n\to\infty}\|x_n - x\|,$$

which contradicts inequality (3.3). Conversely, suppose that X does not satisfy the locally uniform Opial condition. Then there exist a sequence $\{x_n\}$ in X with

$$x_n \rightharpoonup 0, \qquad \limsup_{n\to\infty}\|x_n\| \ge 1,$$

a constant $c > 0$, and a sequence $\{y_m\}$ in X with

$$\|y_m\| \geq c \qquad \text{for all } m \in \mathbb{N},$$

such that

$$1 + \frac{1}{m} > \limsup_{n\to\infty} \|x_n - y_m\| \qquad \text{for all } m \in \mathbb{N}.$$

Hence

$$\limsup_{m\to\infty}\left(\limsup_{n\to\infty} \|x_n - y_m\|\right) \leq 1 \leq \limsup_{n\to\infty} \|x_n\|.$$

By assumption, we have $y_m \to 0$. This contradicts the fact that

$$\|y_m\| \geq c \qquad \text{for all } m \in \mathbb{N}.$$

Therefore, the proof is complete.

3.1 Demiclosedness priniciple

Let C be a nonempty subset of a Banach space X and $T : C \to X$ a mapping. Then T is said to be *demiclosed* at $v \in X$ if for any sequence $\{x_n\}$ in C the following implication holds:

$$x_n \rightharpoonup u \in C \quad \text{and} \quad Tx_n \to v \quad \text{imply} \quad Tu = v.$$

Our first result concerning the demiclosedness principle of nonexpansive mappings is in an Opial space.

Theorem 3.1.1 *Let X be a Banach space that satisfies the Opial condition, C a nonempty weakly compact subset of X, and $T : C \to X$ a nonexpansive mapping. Then the mapping $I - T$ is demiclosed.*

proof: Let $\{x_n\}$ be a sequence in C such that $x_n \rightharpoonup x \in C$ and

$$\lim_{n\to\infty} \|(I-T)x_n - y\| = 0$$

for some $y \in X$. We show that $(I - T)x = y$. Observe that

$$\|x_n - Tx - y\| \leq \|x_n - Tx_n - y\| + \|Tx_n - Tx\|,$$

which implies that

$$\liminf_{n\to\infty} \|x_n - Tx - y\| \leq \liminf_{n\to\infty} \|x_n - x\|.$$

By the Opial condition, we have

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - (Tx + y)\|,$$

a contradiction. Therefore,

$$(I - T)x = y.$$

corollary 3.1.1 *Let X be a reflexive Banach space that satisfies the Opial condition, C a nonempty closed convex subset of X, and $T : C \to X$ a nonexpansive mapping. Then $I - T$ is demiclosed.*

We now extend the demiclosedness principle of nonexpansive mappings in a uniformly convex Banach space without Opial's condition. To do so, we need the following:

Theorem 3.1.2 *Let C be a nonempty closed convex bounded subset of a uniformly convex Banach space X and $T : C \to X$ a nonexpansive mapping. Then $I - T$ is demiclosed on X.*

Proof: Let $\{x_n\}$ be a sequence in C such that $x_n \rightharpoonup x$ and

$$\lim_{n\to\infty} \|x_n - Tx_n - y\| = 0$$

for some $y \in X$. Set

$$T_y x := Tx + y, \qquad x \in C.$$

Then

$$(I - T_y)x_n = (I - T)x_n - y \to 0.$$

If $(I - T_y)x = 0$, then $(I - T)x = y$. Hence, we may assume without loss of generality that $y = 0$. Set

$$\varepsilon_n := \|x_n - Tx_n\|.$$

Because $\varepsilon_n \to 0$ as $n \to \infty$, we may thin out the sequence to make the convergence faster, and we do this in such a way that for each n

$$\varepsilon_n \leq \zeta(\varepsilon_{n-1}) < \varepsilon_{n-1},$$

where $\zeta(\varepsilon)$ for any $\varepsilon > 0$ is the constant Hence for each point

$$z \in \overline{\text{co}}\big(\{x_k : k \geq n\}\big),$$

we have

$$\|z - Tz\| \leq \varepsilon_{n-1}.$$

Because

$$\overline{\text{co}}\big(\{x_k : k \geq n\}\big)$$

is weakly compact (and hence weakly closed) and contains the weak limit x of the sequence $\{x_n\}$, it follows that

$$\|x - Tx\| \leq \varepsilon_{n-1} \longrightarrow 0$$

as $n \to \infty$. Therefore,

$$x = Tx.$$

Chapter 4

EXISTENCE THEOREMS OF CERTAIN NON EXPANSIVE MAPPINGS

4.1 Multivalued nonexpansive mappings

In this section, we consider the problem of solving the operator equation

$$x \in Tx, \tag{4.10}$$

where T is a multivalued nonexpansive mapping in a Banach space.

Definition 4.1.1 *Let C be a nonempty subset of a Banach space X. A mapping $T : C \to CB(X)$ is said to be* nonexpansive *if*

$$H(Tx, Ty) \leq \|x - y\| \qquad \textit{for all } x, y \in C,$$

where $H(\cdot, \cdot)$ is the Hausdorff metric on $CB(X)$.

Recall that the graph $G(A)$ of a multivalued mapping $A : C \to 2^Y$ is

$$G(A) = \{(x, y) \in X \times Y : \ x \in C, \ y \in Ax\},$$

where Y is another Banach space. The mapping A is said to be demiclosed *at $y \in Y$ if*

$$x_n \ (\textit{in } C) \to x \quad \textit{and} \quad y_n \in Ax_n \to y \implies y \in Ax.$$

First, we show that for every compact-valued nonexpansive mapping T, $I-T$ is demiclosed in a Banach space with the Opial condition.

Theorem 4.1.1 *Let C be a nonempty weakly compact subset of a Banach space X with the Opial condition and $T : C \to K(X)$ a nonexpansive mapping. Then $I-T$ is demiclosed.*

Proof: Because the domain of $I-T$ is weakly compact, we must prove that the graph of $I-T$ is only sequentially closed. Let $(x_n, y_n) \in G(I-T)$ be such that $x_n \rightharpoonup x$ and $y_n \to y$. Hence $x \in C$. We now show that $y \in (I-T)x$. Because $y_n \in x_n - Tx_n$, $y_n = x_n - z_n$ for some $z_n \in Tx_n$. By the nonexpansiveness of T, there exists $z_n' \in Tx$ such that

$$\|z_n - z_n'\| \leq H(Tx_n, Tx) \leq \|x_n - x\|. \tag{4.11}$$

It follows from (4.11) that

$$\begin{aligned} \liminf_{n\to\infty} \|x_n - x\| &\geq \liminf_{n\to\infty} \|z_n - z_n'\| \\ &= \liminf_{n\to\infty} \|x_n - y_n - z_n'\|. \end{aligned} \tag{4.12}$$

Because Tx is compact, $z_n' \in Tx$ and $y_n \to y$, there exists a subsequence $\{z_{n_i}'\}$ of $\{z_n'\}$ such that

$$z_{n_i}' \to z \in Tx \quad \text{and} \quad y_{n_i} \to y.$$

Hence from (4.12) we have

$$\liminf_{i\to\infty} \|x_{n_i} - x\| \geq \liminf_{i\to\infty} \|x_{n_i} - y - z\|.$$

By the Opial condition,

$$\liminf_{i\to\infty} \|x_{n_i} - x\| < \liminf_{i\to\infty} \|x_{n_i} - y - z\|,$$

which implies that $y + z = x$. Therefore,

$$y = x - z \in x - Tx.$$

The fixed point theory of multivalued nonexpansive mapping is however much more complicated than the corresponding theory of Single-valued nonexpansive mappings we will concentrate

Theorem 4.1.2 *Let X be a Banach space with the Opial condition, C a nonempty weakly compact convex subset of X, and $T : C \to K(C)$ a nonexpansive mapping. Then T has a fixed point in C.*

Proof: Let u be an element in C and let $\{a_n\}$ be a sequence in $(0, 1)$ such that $\lim_{n\to\infty} a_n = 1$. For each $n \in \mathbb{N}$, define $T_n : C \to K(C)$ by

$$T_n x = (1 - a_n)u + a_n T x, \qquad x \in C. \tag{4.13}$$

Then T_n is a contraction mapping. (Nadler's fixed point theorem), there exists $x_n \in C$ such that $x_n \in T_n x_n$. Because C is weakly compact, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup v \in C$.

From (4.13) we have

$$x_n = (1 - a_n)u + a_n z_n,$$

where $z_n \in T x_n$. Observe that

$$\|x_n - z_n\| = (1 - a_n)\|z_n - u\|.$$

Hence

$$y_n = x_n - z_n \in (I - T)x_n,$$

and

$$y_n \longrightarrow 0.$$

Thus,

$$(x_{n_i}, y_{n_i}) \in G(I - T),$$

and

$$x_{n_i} \rightharpoonup v, \qquad y_{n_i} \longrightarrow 0.$$

It follows from the demiclosedness of $I - T$ at zero that $0 \in (I - T)v$. Therefore, $v \in Tv$.

Proposition 4.1.1 *Let C be a nonempty closed convex subset of a uniformly convex Banach space X and $T : C \to K(C)$ a nonexpansive mapping. Suppose there exists a bounded sequence $\{x_n\}$ in C such that*

$$d(x_n, Tx_n) \longrightarrow 0 \qquad \text{as } n \to \infty.$$

Then T has a fixed point in C.

Proof: We may assume that $\{x_n\}$ is regular and thus asymptotically uniform. Let

$$Z_a(C, \{x_n\}) = \{z\} \quad \text{and} \quad r_a(C, \{x_n\}) = r.$$

Choose $y_n \in Tx_n$ such that

$$\|x_n - y_n\| \longrightarrow 0 \qquad \text{as } n \to \infty.$$

By the compactness of Tz, select $z_n \in Tz$ such that

$$\|y_n - z_n\| \leq H(Tx_n, Tz) \leq \|x_n - z\|. \tag{4.14}$$

Because Tz is compact, there exists a subsequence $\{z_{n_k}\}$ of $\{z_n\}$ such that

$$z_{n_k} \longrightarrow v \in Tz.$$

By the regularity of $\{x_n\}$, we have

$$r_a(C, \{x_{n_k}\}) = r_a(C, \{x_n\}) = r,$$

and

$$Z_a(C, \{x_n\}) = Z_a(C, \{x_{n_k}\}) = \{z\}.$$

Because

$$\|x_{n_k} - v\| \leq \|x_{n_k} - y_{n_k}\| + \|y_{n_k} - z_{n_k}\| + \|z_{n_k} - v\|,$$

it follows from (5.34) that

$$\limsup_{k\to\infty} \|x_{n_k} - v\| \leq r,$$

and hence $Z_a(C, \{x_n\}) = z = v$.

Therefore, $z \in Tz$, and so z is a fixed point of T.

4.2 Asymptotically nonexpansive mappings

Let C be a nonempty subset of a Banach space X. A mapping $T : C \to C$ is said to be asymptotically nonexpansive if for each $n \in \mathbb{N}$, there exists a positive constant $k_n \geq 1$ with

$$\lim_{n\to\infty} k_n = 1$$

such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\|, \qquad \text{for all } x, y \in C.$$

The following example shows that the class of asymptotically nonexpansive mappings is essentially wider than the class of nonexpansive mappings.

Definition 4.2.1 *Let C be a nonempty subset of a normed space X and $T : C \to C$ a mapping. Then T is said to be*

1. *asymptotically regular at $x_0 \in C$ if*

$$\lim_{n\to\infty} \|T^n x_0 - T^{n+1} x_0\| = 0;$$

2. *weakly asymptotically regular at $x_0 \in C$ if*

$$T^n x_0 - T^{n+1} x_0 \rightharpoonup 0;$$

3. *asymptotically regular on C if for any $x \in C$,*

$$\lim_{n\to\infty} \|T^n x - T^{n+1} x\| = 0;$$

4. *uniformly asymptotically regular if*

$$\lim_{n\to\infty} \left(\sup_{x\in C} \|T^n x - T^{n+1} x\| \right) = 0;$$

5. *reasonable wanderer in C if starting at any $x \in C$,*

$$\sum_{n=0}^{\infty} \|T^n x - T^{n+1} x\| < \infty.$$

Note that every uniformly asymptotically regular mapping is asymptotically regular.

Remarks 4.2.1 *The asymptotic regularity of a mapping T at a point x_0 implies the existence of an approximating fixed point sequence of that mapping, but the converse is not true.*

Example 4.2.1 *Let B_H be the closed unit ball in the Hilbert space $H = \ell_2$ and $T : B_H \to B_H$ a mapping defined by*

$$T(x_1, x_2, x_3, \ldots) = (0, x_1^2, a_2x_2, a_3x_3, \ldots),$$

where $\{a_i\}$ is a sequence of real numbers such that $0 < a_i < 1$ and

$$\prod_{i=2}^{\infty} a_i = \frac{1}{2}.$$

Then

$$\|Tx - Ty\| \leq 2\|x - y\| \quad \text{for all } x, y \in B_H,$$

i.e., T is Lipschitzian, but not nonexpansive. Observe that

$$\|T^n x - T^n y\| \leq 2\prod_{i=2}^{n} a_i \,\|x - y\| \quad \text{for all } x, y \in B_H \text{ and } n \geq 2.$$

Here

$$k_n = 2\prod_{i=2}^{n} a_i \longrightarrow 1 \quad \text{as } n \to \infty.$$

Therefore, T is asymptotically nonexpansive, but not nonexpansive. There is also a connection between the demiclosedness principle and the fixed point theory of asymptotically nonexpansive mappings. Some simple results concerning the demiclosedness principle of asymptotically nonexpansive mappings are given in the following theorems.

Proposition 4.2.1 *Let C be a nonempty closed convex subset of a uniformly convex Banach space X and $T : C \to C$ an asymptotically nonexpansive mapping. If $\{y_n\}$ is a bounded sequence in C such that*

$$\lim_{n\to\infty} \|y_n - Ty_n\| = 0$$

and

$$Z_a(C, \{y_n\}) = \{v\},$$

then v is a fixed point in C.

Proof: We define a sequence $\{z_m\}$ in C by

$$z_m = T^m v, \qquad m \in \mathbb{N}.$$

For integers $m, n \in \mathbb{N}$, we have

$$\begin{aligned} \|z_m - y_n\| &\leq \|T^m v - T^m y_n\| + \|T^m y_n - T^{m-1} y_n\| + \cdots + \|Ty_n - y_n\| \\ &\leq k_m \|v - y_n\| + \left(\|Ty_n - y_n\| + \sum_{i=1}^{m-1} k_i \|y_n - Ty_n\| \right). \end{aligned} \tag{4.21}$$

Then by (4.21) we have

$$r_a(z_m, \{y_n\}) = \limsup_{n\to\infty} \|y_n - z_m\| \leq k_m r_a(v, \{y_n\}) = k_m r_a(C, \{y_n\}).$$

Hence

$$|r_a(z_m, \{y_n\}) - r_a(C, \{y_n\})| \leq (k_m - 1)\, r_a(C, \{y_n\}) \longrightarrow 0 \quad \text{as } m \to \infty.$$

It follows that

$$T^m v \longrightarrow v.$$

By the continuity of T, we have

$$Tv = T\Big(\lim_{m\to\infty} T^m v\Big) = \lim_{m\to\infty} T^{m+1} v = v.$$

Theorem 4.2.1 *Let X be a uniformly convex Banach space with the Opial condition, C a nonempty closed convex (not necessarily bounded) subset of X, and $T : C \to C$ an asymptotically nonexpansive mapping. Then $I - T$ is demiclosed at zero.*

Theorem 4.2.2 *Let X be a Banach space with the locally uniform Opial condition, C a nonempty weakly compact convex subset of X, and $T : C \to C$ an asymptotically nonexpansive mapping. Then $I - T$ is demiclosed at zero.*

Theorem 4.2.3 *Let C be a nonempty closed convex bounded subset of a uniformly convex Banach space X and $T : C \to C$ an asymptotically nonexpansive mapping. Then $I - T$ is demiclosed at zero.*

Proof. Let $\{x_n\}$ be a sequence in C such that $x_n \rightharpoonup x \in C$ and $x_n - Tx_n \to 0$ as $n \to \infty$. We show that $T^n x \to x$. Indeed, because $\{x_n\}$ is weakly convergent to x, there exists for each integer $n \in \mathbb{N}$ a convex combination

$$y_n = \sum_{i=1}^{m(n)} t_i^{(n)} x_{i+n}, \qquad \left(t_i^{(n)} \geq 0 \text{ and } \sum_{i=1}^{m(n)} t_i^{(n)} = 1 \right),$$

such that$\|y_n - x\| < \frac{1}{n}$. For an arbitrary but fixed $j \in \mathbb{N}$,because $(I-T)x_n \to 0$, there is an $n_0 = n_0(\varepsilon, j)$ so large that $\frac{1}{n_0} < \varepsilon$ and $\|(I - T^j)x_n\| < \varepsilon, \qquad$ for $n \geq n_0$. Because X is uniformly convex, there exists a strictly increasing convex and continuous function $g : \mathbb{R}^+ \to \mathbb{R}^+$, $g(0) = 0$ such that for any nonexpansive mapping $S : C \to X$, for any finite many elements $\{u_i\}_{i=1}^n$ in C, and for any finite many nonnegative numbers $\{t_i\}_{i=1}^n$ with $\sum_{i=1}^n t_i = 1$, the following inequality holds:

$$g\left(\left\| S\left(\sum_{i=1}^n t_i u_i \right) - \sum_{i=1}^n t_i S u_i \right\| \right) \leq \max_{1 \leq i,j \leq n} \left(\|u_i - u_j\| - \|Su_i - Su_j\| \right). \tag{4.22}$$

Suppose L_j is the Lipschitz constant of T^j. Then $L_j^{-1} T^j$ is nonexpansive in C, and it follows from (4.22) that

$$\begin{aligned}
\|T^j y_n - y_n\| &\leq \left\| T^j y_n - \sum_{i=1}^{m(n)} t_i^{(n)} T^j x_{i+n} \right\| + \left\| \sum_{i=1}^{m(n)} t_i^{(n)} T^j x_{i+n} - y_n \right\| \\
&\leq \left\| T^j \left(\sum_{i=1}^{m(n)} t_i^{(n)} x_{i+n} \right) - \sum_{i=1}^{m(n)} t_i^{(n)} T^j x_{i+n} \right\| + \sum_{i=1}^{m(n)} t_i^{(n)} \|T^j x_{i+n} - x_{i+n}\| \\
&\leq L_j g^{-1} \left(\max_{1 \leq i,k \leq m(n)} \left(\|x_{i+n} - x_{k+n}\| - L_j^{-1} \|T^j x_{i+n} - T^j x_{k+n}\| \right) \right) + \varepsilon \\
&\leq L_j g^{-1} \left(2\varepsilon + (1 - L_j^{-1}) \operatorname{diam}(C) \right) + \varepsilon.
\end{aligned} \tag{4.23}$$

because

$$\begin{aligned}
&\|x_{i+n} - x_{k+n}\| - L_j^{-1} \|T^j x_{i+n} - T^j x_{k+n}\| \\
&\leq \|x_{i+n} - T^j x_{i+n}\| + \|x_{k+n} - T^j x_{k+n}\| \\
&\qquad + (1 - L_j^{-1}) \|T^j x_{i+n} - T^j x_{k+n}\| \\
&\leq 2\varepsilon + (1 - L_j^{-1}) \operatorname{diam}(C).
\end{aligned}$$

Taking the limit superior as $n \to \infty$ in (4.23), we obtain

$$\limsup_{n\to\infty} \|T^j y_n - y_n\| \le L_j g^{-1}\left(2\varepsilon + (1 - L_j^{-1})\operatorname{diam}(C)\right) + \varepsilon \quad \text{for all } j \in \mathbb{N}. \quad (4.24)$$

For each $j \in \mathbb{N}$, we have

$$\|T^j x - x\| \le \|T^j x - T^j y_n\| + \|T^j y_n - y_n\| + \|y_n - x\| \le (L_j + 1)\|y_n - x\| + \|T^j y_n - y_n\|. \quad (4.25)$$

Because $y_n \to x$, it follows from (4.24) and (4.25) that

$$\begin{aligned}\|T^j x - x\| &\le \limsup_{n\to\infty}\left((L_j+1)\|y_n - x\| + \|T^j y_n - y_n\|\right)\\ &\le L_j g^{-1}\left(2\varepsilon + (1 - L_j^{-1})\operatorname{diam}(C)\right) + \varepsilon\\ &= g^{-1}(0) = 0, \qquad \text{as } j \to \infty \text{ and } \varepsilon \to 0.\end{aligned}$$

This shows that $T^n x \to x$ and by the continuity of T, we obtain that $x = Tx$.

Theorem 4.2.4 (Goebel and Kirk's Fixed Point Theorem) *Let C be a nonempty closed convex bounded subset of a uniformly convex Banach space X and $T : C \to C$ an asymptotically nonexpansive mapping. Then T has a fixed point in C.*

Proof. For fixed $y \in C$ and $r > 0$, set
$R_y := \{r :$ there exists $k \in \mathbb{N}$ with $C \cap (\bigcap_{i=k}^{\infty} B_r[T^i y]) \neq \emptyset\}$, and $d := \operatorname{diam}(C)$.

Then $d \in R_y$. Hence $R_y \neq \emptyset$. Let $r_0 = \inf\{r : r \in R_y\}$. For each $\varepsilon > 0$, we define

$$C_\varepsilon = \bigcup_{k=1}^{\infty}\left(\bigcap_{i=k}^{\infty} B_{r_0+\varepsilon}[T^i y]\right).$$

Thus, for each $\varepsilon > 0$, the set $C_\varepsilon \cap C$ is nonempty and convex. The reflexivity of X implies that

$$\bigcap_{\varepsilon>0}\left(\overline{C_\varepsilon} \cap C\right) \neq \emptyset.$$

Note that for $x \in \bigcap_{\varepsilon>0}(\overline{C_\varepsilon} \cap C)$ and $\eta > 0$, there exists an integer n_0 such that

$$\|x - T^n y\| \le r_0 + \eta, \qquad \text{for all } n \ge n_0.$$

Now, let $x \in \bigcap_{\varepsilon>0}(\overline{C_\varepsilon} \cap C)$ and suppose, for contradiction, that the sequence $\{T^n x\}$ does not converge strongly to x.Then there exist $\varepsilon > 0$ and a subsequence $\{T^{n_i} x\}$ of $\{T^n x\}$ such that

$$\|T^{n_i} x - x\| \ge \varepsilon, \qquad \text{for all } i = 1, 2, \ldots.$$

Suppose k_n is the Lipschitz constant of T^n. Then for $m > n$, we have

$$\|T^n x - T^m x\| \leq k_n \|x - T^{m-n} x\|.$$

Assume that $r_0 > 0$ and choose $\alpha > 0$ such that

$$\left(1 - \delta_X \left(\frac{\varepsilon}{r_0 + \alpha}\right)\right)(r_0 + \alpha) < r_0.$$

Select n such that

$$\|x - T^n x\| \geq \varepsilon \quad \text{and} \quad k_n \left(r_0 + \frac{\alpha}{2}\right) \leq r_0 + \alpha.$$

If $n_0 \geq n$ is sufficiently large, then $m > n_0$ implies

$$\|x - T^{m-n} y\| \leq r_0 + \frac{\alpha}{2}.$$

Because

$$\|T^n x - T^m y\| \leq k_n \|x - T^{m-n} y\| \leq k_n \left(r_0 + \frac{\alpha}{2}\right) \leq r_0 + \alpha$$

and

$$\|x - T^m y\| \leq r_0 + \alpha,$$

it follows from the uniform convexity of X that for $m > n_0$,

$$\left\|\frac{1}{2}(x + T^n x) - T^m y\right\| \leq \left(1 - \delta_X \left(\frac{\varepsilon}{r_0 + \alpha}\right)\right)(r_0 + \alpha) < r_0.$$

This contradicts the definition of r_0. Hence we conclude that $r_0 = 0$ or $x = Tx$. But $r_0 = 0$ implies that $\{T^m y\}$ is a Cauchy sequence and hence $\lim_{m\to\infty} T^m y = x = Tx$. Therefore, the set $\bigcap_{\varepsilon>0}\left(\overline{C_\varepsilon} \cap C\right)$ is a singleton that is a fixed point of T.

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