A STUDY ON FIXED POINT THEOREMS USING PICARD AND KRASNOSELSKIJ ITERATIONS

A Dissertation submitted to University of Kerala in partial fulfillment of the requirements for the award of degree of

MASTER OF SCIENCE IN MATHEMATICS

Name	**:**	**FATHIMA A**
Candidate Code	**:**	**62024142008**
Exam Code	**:**	**62023402**
Subject Code	**:**	**MM 545**

DEPARTMENT OF MATHEMATICS
T.K.M. COLLEGE OF ARTS AND SCIENCE
KOLLAM - 691005

2024-2026

CERTIFICATE

This is to certify that the dissertation entitled **“A STUDY ON FIXED POINT THEOREMS USING PICARD AND KRASNOSELSKIJ ITERATIONS”** is a record of studies carried out by **FATHIMA A**, M.Sc. Mathematics student of the year **2024-2026** at **T.K.M. College of Arts and Science**, Kollam , under my guidance and submitted to the University Of Kerala in partial fulfilment of the Degree of Master of Science in Mathematics.

Kollam
July 2026

Mr.HARILAL N THAZHIKKATTUSERIL	**Dr.ASWATHY M R**
Assistant Professor	H.O.D. and Assistant Professor
Department of Mathematics	Department of Mathematics
T.K.M. College of Arts and Science	T.K.M. College of Arts and Science

DECLARATION

I hereby declare that the dissertation entitled **"A STUDY ON FIXED POINT THEOREMS USING PICARD AND KRASNOSELSKIJ ITERATIONS"** is a record of studies carried out by me under the guidance of **Mr. HARILAL N THAZHIKKATTUSERIL** , Assistant Professor, Department of Mathematics, T.K.M. College of Arts and Science, Kollam and has not been submitted by me previously for the award of any degree.

Kollam **FATHIMA A**

July 2026

ACKNOWLEDGEMENT

I would like to record my heartfelt gratitude to my guide, **Mr. HARILAL N THAZHIKKATTUSERIL**, Assistant Professor, Department of Mathematics, T.K.M. College of Arts and Science, Kollam, for his guidance and support, which enabled me to complete this work successfully. I express my sincere thanks to **Dr. ASWATHY M R**, Head of the Department of Mathematics, and the Principal,T.K.M. College of Arts and Science,Kollam for their valuable support and cooperation throughout the course of this project. I express my deepest gratitude to my beloved parents, for their unconditional love, prayers, sacrifices, and unwavering moral support throughout my academic journey .I also extend my heartfelt thanks to my friends for the unwavering support, encouragement, and reassuring presence throughout the completion of this project.Above all, I humbly express my heartfelt gratitude to Almighty God for his abundant blessings, guidance, strength, and unseen yet unfailing presence, which sustained me throughout the successful completion of this work.

Kollam **FATHIMA A**

July 2026

Contents

INTRODUCTION

Fixed point theory is one of the most important and rapidly developing branches of modern mathematics. It deals with the study of points that remain unchanged under a given mapping and has become a fundamental tool in various areas of pure and applied mathematics. Over the past century, fixed point theory has evolved from a theoretical discipline into a powerful framework for solving a wide range of mathematical problems. It forms a rich blend of analysis, topology, and geometry and serves as a bridge between pure mathematics and its applications.

The significance of fixed point theory lies in its extensive range of applications. Fixed point results play a crucial role in the study of differential equations, integral equations, variational inequalities, optimization problems, approximation theory, and dynamical systems. Beyond mathematics, fixed point methods have found applications in economics, engineering, computer science, biology, physics, statistics, and many other scientific disciplines. Concepts such as equilibrium states in economics, Nash equilibria in game theory, fluid flow problems, control systems, and population dynamics can often be formulated and analyzed using fixed point techniques. Thus, fixed point theory has become an indispensable tool in both theoretical investigations and practical applications.

Let X be a nonempty set and let $T : X \to X$ be a self-mapping. A point $x \in X$ is called a *fixed point* of T if

$$T(x) = x.$$

In other words, a fixed point is a point that remains invariant under the action of the mapping. Depending on the nature of the mapping, a fixed point may be unique, multiple, infinitely many, or may not exist at all. Therefore, one of the central objectives of fixed point theory is to establish conditions under which a mapping admits a fixed point and to investigate the properties of such points.

While existence theorems guarantee the presence of fixed points, it is equally important to determine how these fixed points can be obtained in practice. In many situations, finding an exact fixed point is difficult or impossible. This has led to the development of iterative methods, which generate sequences converging to fixed points. Iterative techniques provide constructive procedures for approximating fixed points and have become an essential component of fixed point theory. Among the various iterative schemes proposed in the literature, the Picard iteration is one of the earliest and most fundamental methods. Introduced in connection with differential equations, the Picard iterative process generates a sequence by repeatedly applying the mapping to an initial point. Under suitable contractive conditions, this process converges to the unique fixed point of the mapping. Due to its simplicity and effectiveness, the Picard iteration has become a classical tool in fixed point theory.

Another important iterative method is the Krasnoselkij iteration, which combines the current iterate and its image under the mapping through a convex combination. This process has been extensively studied for approximating fixed points of nonexpansive and other classes of nonlinear mappings. The Krasnoselskij iteration often provides greater flexibility and wider applicability than the Picard iteration, particularly in situations where the direct application of Picard iteration is not feasible.

The study of iterative processes is motivated by the need to obtain efficient approximation methods for fixed points. New iterative schemes are frequently introduced either to handle broader classes of mappings or to improve the rate of convergence of existing methods. Consequently, the comparison and analysis of different iterative procedures have become important topics of research in fixed point theory. The primary objective of this project is to study fixed point theorems and the approximation of fixed points using the Picard and Krasnoselskij iterative processes. The project begins with the basic concepts of metric spaces, normed spaces, and iterative methods that are required for the study of fixed point theory. It then discusses the Picard iteration together with important fixed point results, including Banach's Fixed Point Theorem, the Nemytzki–Edelstein Theorem, quasi-nonexpansive operators, and Maia's Fixed Point Theorem. Finally, the Krasnoselskij iteration is studied for various classes of nonlinear operators such as nonexpansive, strictly pseudocontractive, generalized pseudocontractive, pseudo φ-contractive, and quasi-nonexpansive operators. The convergence properties of these iterative methods are examined, highlighting their significance in the approximation of fixed points.

Chapter 1

Preliminaries

Definition 1.0.1. Let X be a non-empty set. A mapping

$$d : X \times X \to \mathbb{R}^+$$

is called a *metric* or a *distance* on X if

(d$_1$) $d(x, y) = 0 \Leftrightarrow x = y$; (*"separation axiom"*)

(d$_2$) $d(y, x) = d(x, y)$, for all $x, y \in X$; (*"symmetry"*)

(d$_3$) $d(x, z) \leq d(x, y) + d(y, z)$, for all $x, y, z \in X$ (*"the triangle inequality"*).

A set X endowed with a metric d is called a *metric space* and is denoted by (X, d).

Definition 1.0.2. Let (X, d) be a metric space. The topology having as basis the family of all open balls

$$B(x; r), \quad x \in X, \ r > 0,$$

is called the *topology* induced by the metric d.

Definition 1.0.3. A metric space (X, d) is called *complete* if every Cauchy sequence in X is convergent.

Definition 1.0.4. Let (X, d) be a metric space. A mapping $T : X \to X$ is called:

(C1) *Lipschitzian* (or L - *Lipschitzian*) if there exists $L > 0$

$$d(Tx, Ty) \leq L\, d(x, y), \qquad \forall\, x, y \in X.$$

(C2) *(Strict) contraction* (or a-*contraction*) if T is a-Lipschitzian, with $a \in [0, 1)$.

(C3) *Nonexpansive* if T is 1-Lipschitzian.

(C4) *Contractive* if

$$d(Tx, Ty) < d(x, y), \qquad \forall\, x, y \in X,\ x \neq y.$$

(C5) *Isometry* if

$$d(Tx, Ty) = d(x, y), \qquad \forall\, x, y \in X.$$

Theorem 1.0.1 (Contraction Mapping Principle). *Let (X, d) be a complete metric space and let $T : X \to X$ be a given contraction. Then T has a unique fixed point p, and*

$$T^n(x) \to p \qquad \text{as } n \to \infty,$$

for each $x \in X$.

Definition 1.0.5. Let E be a real (or complex) vector space. A *norm* on E is a mapping

$$\|\cdot\| : E \to \mathbb{R}^+$$

having the following properties:

(n1)

$$\|x\| = 0 \iff x = 0,$$

where 0 is the null element of E;

(n2)

$$\|\lambda x\| = |\lambda|\, \|x\|,$$

for any $x \in E$ and any scalar λ;

(n3)

$$\|x + y\| \leq \|x\| + \|y\|,$$

for all $x, y \in E$ (the triangle inequality).

The pair $(E, \|\cdot\|)$ is called a *normed (linear) space*.

Definition 1.0.6. A *Banach space* is a complete normed linear space. In other words, a normed linear space $(E, \|\cdot\|)$ is called a *Banach space* if every Cauchy sequence $\{x_n\}$ in E converges to some element $x \in E$.

Definition 1.0.7. A Banach space $(E, \|\cdot\|)$ is said to be *uniformly convex* if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that whenever $x, y \in E$ satisfy

$$\|x\| \leq 1, \qquad \|y\| \leq 1, \qquad \|x - y\| \geq \varepsilon,$$

then

$$\left\| \frac{x+y}{2} \right\| < 1 - \delta.$$

Theorem 1.0.2. *Let C be a closed, bounded, and convex subset of a uniformly convex Banach space and $T : C \mapsto C$ a nonexpansive map. Then T has a fixed point.*

Definition 1.0.8. A normed linear space E is called *strictly convex* if $x, y \in E$ with

$$\|x\| = \|y\| = 1$$

and

$$\|(1-\lambda)x + \lambda y\| = 1$$

for some $\lambda \in (0, 1)$, holds if and only if $x = y$.

Definition 1.0.9. Let E^* be the dual space of a real Banach space E. The multivalued mapping $J : E \to P(E^*)$ defined by

$$Jx = \{f \in E^* : \langle f, x\rangle = \|x\| \, \|f\|, \quad \|x\| = \|f\|\}$$

is called the *normalized duality mapping* of E.

Definition 1.0.10. Let E be an arbitrary real Banach space. A mapping T with domain $D(T)$ and range $R(T)$ in E is called

(a) *strongly pseudocontractive* if there exists $k > 0$ such that for all $x, y \in D(T)$ there exists $j(x-y) \in J(x-y)$ satisfying

$$\langle (I-T)x - (I-T)y, \ j(x-y)\rangle \geq k\|x-y\|^2; \tag{1.1}$$

(b) *pseudocontractive* if for each $x, y \in D(T)$ there exists $j(x-y) \in J(x-y)$ such that

$$\langle (I-T)x - (I-T)y,\ j(x-y)\rangle \geq 0, \tag{1.2}$$

where J is the normalized duality mapping.

Definition 1.0.11. Let X be a nonempty set and $T : X \to X$ a selfmap. We say that $x \in X$ is a fixed point of T if

$$T(x) = x$$

and denote by F_T or $\mathrm{Fix}(T)$ the set of all fixed points of T.

Example 1.0.1.

1. If $X = \mathbb{R}$ and $T(x) = x^2 + 5x + 4$, then

$$F_T = \{-2\}.$$

2. If $X = \mathbb{R}$ and $T(x) = x^2 - x$, then

$$F_T = \{0, 2\}.$$

1.1 Iterative Processes

The development of iterative processes has greatly enriched fixed point theory and its applications. Various iterative schemes have been introduced to approximate fixed points of different classes of mappings. Among these, the Picard, Mann, Ishikawa, Noor, and Krasnosel'skiĭ iterations are some of the most widely studied methods. These processes provide constructive techniques for obtaining fixed points and have found applications in many areas of mathematics and applied sciences. The study of iterative processes naturally leads to the following question: if the existence of a fixed point of an operator is guaranteed, then how can that fixed point be obtained? In general, fixed points of linear and nonlinear mappings can be determined by either direct methods or iterative methods. However, direct methods are not always applicable, and in such situations iterative methods provide an effective alternative for approximating fixed points. As a result, iterative approximation techniques have become an important tool in fixed point theory and its applications. New iterative methods are usually introduced for two main reasons:

1. To approximate fixed points in situations where existing iterative processes fail.
2. To obtain a faster rate of convergence than previously known iterative methods.

Chapter 2

Picard Iteration

Let X be a non empty set and $T : X \mapsto X$ be a selfmap . We say that $x \in X$ is a fixed point of T if

$$T(x) = x$$

and denote by F_T or Fix the set of all fixed points of T.

$$x_n = T(x_n - 1) = T^n(x_0) \qquad\qquad n = 1, 2, ..$$

Let X be any set and $T : X \to X$ a self-map. For any given $x \in X$, we define $T^n(x)$ inductively by

$$T^0(x) = x$$

and

$$T^{n+1}(x) = T(T^n(x)).$$

We call $T^n(x)$ the nth iterate of x under T. In order to simplify the notations we will often use Tx instead of $T(x)$. The mapping T^n $(n \geq 1)$ is called the nth iterate of T. For any $x_0 \in X$, the sequence $\{x_n\}_{n\geq 0} \subset X$ given by

$$x_n = Tx_{n-1} = T^n x_0, \qquad n = 1, 2, \ldots$$

is called *the sequence of successive approximations with the initial value* x_0 or the *Picard iteration* starting at x_0 .

2.1 Banach's Fixed Point Theorem

Theorem 2.1.1. *Let (X,d) be a complete metric space and* $T : X \mapsto X$ *be an a-contraction , that is an operator satisfying*

$$d(Tx, Ty) \leq ad(x, y) \text{ ,for any } x, y \in X$$

with $a \in [0, 1)$ *fixed.Then*

(i) T has a unique fixed point , that is , $F_T = x*$ *;*

(ii) The Picard iteration associated to T, i,e .,the sequence $\{x_n\}_{n=0}^{\infty}$ *, defined by*

$$x_n = T(x_n - 1) = T^n(x_0), n = 1, 2, ...,$$

converges to x^* *, for any initial* $x_0 \in X$*;*

(iii) The following a priori and a posteriori error estimates hold:

$$d(x_n, x^*) \leq \frac{a^n}{1-a} d(x_0, x_1), n = 0, 1, 2, ...$$

$$d(x_n, x^*) \leq \frac{a^n}{1-a} 1 - ad(x_n - 1, x_n), n = 0, 1, 2, ...$$

(iv) The rate of convergence is given by

$$d(x_n, x^*) \leq d(x_n - 1, x*) \leq a^n d(x_0, x^*), n = 1, 2, ...$$

Proof : Assume that $x^*, y^* \in F_T$ and $x^* \neq y^*$. Since $0 \leq a < 1$, we obtain the contradiction

$$d(x^*, y^*) = d(Tx^*, Ty^*) \leq a\, d(x^*, y^*) < d(x^*, y^*).$$

To prove the existence of a fixed point, we show that for any $x_0 \in X$, the Picard iteration $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence.

$$d(x_2, x_1) = d(Tx_1, Tx_0) \leq a\, d(x_1, x_0),$$

and by induction,

$$d(x_{n+1}, x_n) \leq a^n d(x_1, x_0), \qquad n = 0, 1, 2, \ldots$$

Thus, for any $n, p \in \mathbb{N}$ with $p > 0$,

$$
\begin{aligned}
d(x_{n+p}, x_n) &\le \sum_{k=n}^{n+p-1} d(x_{k+1}, x_k) \\
&\le \sum_{k=n}^{n+p-1} a^k d(x_1, x_0) \\
&\le \frac{a^n}{1-a}\, d(x_1, x_0).
\end{aligned}
\tag{1}
$$

Since $0 \le a < 1$, we have $a^n \to 0$ as $n \to \infty$. Together with (1), this shows that $\{x_n\}$ is a Cauchy sequence. Since (X, d) is complete, there exists $x^* \in X$ such that

$$x_n \to x^*.$$

Since every Lipschitz mapping is continuous,

$$x^* = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} T(x_n) = T\Big(\lim_{n\to\infty} x_n\Big) = Tx^*.$$

Hence x^* is a fixed point of T.

Therefore, for every $x_0 \in X$, the Picard iteration converges to a fixed point of T. Since T has at most one fixed point, the limit is independent of the choice of x_0 and equals the unique fixed point x^*.

To prove (iii), we have

$$d(x_{n+p}, x_n) \le \frac{a^n}{1-a}\, d(x_0, x_1), \qquad \forall\, p \in \mathbb{N}^*.$$

By continuity of the metric, letting $p \to \infty$ yields

$$
\begin{aligned}
d(x_n, x^*) &= d(x^*, x_n) \\
&= \lim_{p\to\infty} d(x_{n+p}, x_n) \\
&\le \frac{a^n}{1-a}\, d(x_0, x_1), \qquad n \ge 0.
\end{aligned}
$$

To obtain the *a posteriori* estimation , let us notice that, by (1), we have

$$d(x_{n+1}, x_n) \le a\, d(x_n, x_{n-1}),$$

and, by induction,

$$d(x_{n+k}, x_{n+k-1}) \le a^k d(x_n, x_{n-1}), \qquad k \in \mathbb{N}^*,$$

so

$$d(x_{n+p}, x_n) \leq \left(a + a^2 + \cdots + a^p\right) d(x_n, x_{n-1})$$
$$\leq \frac{a}{1-a}\, d(x_n, x_{n-1}).$$

By letting $p \to \infty$ in the last inequality, we obtain the posteriori estimation.

Definition 2.1.1. Let (X,d) be a complete metric space . A mapping $T : X \mapsto X$ is called *(strict) Picard mapping* if there exists $x^* \in X$ such that $F_T = x^*$ and $T^n(x_0) \mapsto x^*$ uniformly for all $x_0 \in X$.

Remarks 2.1.1. If (X,d) is a complete metric space , then any contraction $T : X \mapsto X$ is a Picard mapping.

2.2 Theorem of Nemytzki-Edelstein

The Banach contraction principle requires the existence of a fixed contraction constant $a \leq 1$.Nemytzkii and Edelstein weakened this requirement by replacing the contraction condition with a more general contractive condition.

Theorem 2.2.1. *Let (X,d) be a compact metric space and $T : X \mapsto X$ be a contractive operator , then T is a strict Picard operator.*

Proof : A metric space is compact if and only if every family of closed subsets of X with finite intersection property (i.e., any finite number of sets in the family has a nonempty intersection) has a nonempty intersection. From the contractiveness of T we have card $F_T \leq 1$.

Let $x_0 \in X$ and $x_n + 1 = (1-\lambda)x_n + \lambda x_n$,$n = 0, 1, 2, ...$, $x_n = T^n x_0$, $n \geq 0$, be the Picard iteration associated to T.

Since (X, d) is compact, it results that there exists a subsequence $\{x_{n_k}\}_{k=0}^{\infty}$ of $\{x_n\}_{n=0}^{\infty}$ such that $\{x_{n_k}\}_{k=0}^{\infty}$ converges to a certain $x^* \in X$ as n tends to ∞. As T is contractive, we deduce that T is continuous and that the sequence $\{d(x_n, x_{n+1})\}_{n=0}^{\infty}$ has strictly decreasing positive terms and hence is convergent.

Then, using the continuity of the metric, we have

$$\lim_{k\to\infty} d(x_{n_k}, Tx_{n_k}) = d(x^*, Tx^*),$$

therefore

$$d(x^*, Tx^*) = \lim_{n\to\infty} d(x_n, x_{n+1}) = \lim_{n\to\infty} d(x_{n+1}, x_{n+2}) = d(Tx^*, T^2x^*).$$

If $x^* \neq Tx^*$, then from the contractive condition we get the contradiction

$$d(x^*, Tx^*) = d(Tx^*, T(Tx^*)) < d(x^*, Tx^*).$$

Consequently,

$$x^* = Tx^* ,$$

i.e.,

$$F_T = \{x^*\}.$$

This shows that for any $x_0 \in X$, the Picard iteration converges in X and its limit is the unique fixed point of T.

Remarks 2.2.1. For a contractive operator ,we generally have no information about the convergence rate of Picard iteration.

2.3 Quasi-Nonexpansive Operators

In the previous sections, fixed point results were obtained for various classes of contractive mappings. The following theorem shows that a Picard operator need not be continuous. This observation motivates the study of more general classes of mappings. One such important class is that of quasi-nonexpansive operators, which generalize nonexpansive mappings while retaining useful fixed point properties.

Theorem 2.3.1. *Let (X, d) be a complete metric space and let $T : X \to X$ be a mapping for which there exists $a \in \left[0, \frac{1}{2}\right)$ such that*

$$d(Tx, Ty) \leq a[d(x, Tx) + d(y, Ty)], \qquad \forall x, y \in X.$$

Then T is a Picard operator.

Remarks 2.3.1. establishes that a mapping satisfying the above contractive-type condition is a Picard operator. Hence, it possesses a unique fixed point and the Picard iteration converges to this fixed point from any initial approximation. The significance of this result lies in the fact that continuity of the operator is not required.

Example 2.3.1. Let $X = \mathbb{R}$ and define $T : X \to X$ by

$$T(x) = \begin{cases} 0, & x \leq 2, \\ -\dfrac{1}{2}, & x > 2. \end{cases}$$

Then:

1. T is not continuous.
2. T satisfies the condition of Theorem 2.3.1 with $a = \dfrac{1}{5}$ and hence is a Picard operator.
3. T is not nonexpansive.

This example shows that a mapping may be discontinuous and non-nonexpansive, yet still be a Picard operator. Therefore, the class of Picard operators is broader than the classes of continuous and nonexpansive mappings.

Corollary 2.3.1. *Let the assumptions of Theorem 2.3.1 be satisfied. Then the error estimates for the Picard iteration are given by*

$$d(x_n, x^*) \leq \frac{\alpha^n}{1-\alpha}\, d(x_0, x_1), \qquad n = 0, 1, 2, \ldots,$$

and

$$d(x_n, x^*) \leq \frac{\alpha}{1-\alpha}\, d(x_n, x_{n-1}), \qquad n = 0, 1, 2, \ldots,$$

where

$$\alpha = \frac{a}{1-a}.$$

The above estimates provide bounds for the approximation error in the Picard iteration. They indicate how close an iterate is to the fixed point and provide information about the rate of convergence.

Definition 2.3.1. An operator $T : X \to X$ is said to be *quasi-nonexpansive* if T has at least one fixed point and

$$d(Tx, p) \leq d(x, p), \qquad \forall x \in X,$$

for every fixed point $p \in F(T)$.

Remarks 2.3.2. Every contractive mapping is quasi-nonexpansive. However, the converse is not necessarily true. Thus, quasi-nonexpansive operators form a broader class of mappings than contractions.

Theorem 2.3.2 (Zamfirescu's Theorem)**.** *Let (X, d) be a complete metric space and let $T : X \to X$ be a mapping for which there exist real numbers α, β and γ satisfying*

$$0 \leq \alpha < 1, \qquad 0 \leq \beta < \frac{1}{2}, \qquad 0 \leq \gamma < \frac{1}{2},$$

such that for every $x, y \in X$, at least one of the following conditions holds:

(z_1)

$$d(Tx, Ty) \leq \alpha d(x, y);$$

(z_2)

$$d(Tx, Ty) \leq \beta[d(x, Tx) + d(y, Ty)];$$

(z_3)

$$d(Tx, Ty) \leq \gamma[d(x, Ty) + d(y, Tx)].$$

Then T is a Picard operator.

Proof : We first fix $x, y \in X$. At least one of (z_1), (z_2) or (z_3) is true.
If (z_2) holds, then we have

$$d(Tx, Ty) \leq \beta \left[d(x, Tx) + d(y, Ty)\right]$$

$$\leq \beta \left\{d(x, Tx) + \left[d(y, x) + d(x, Tx) + d(Tx, Ty)\right]\right\}.$$

So

$$(1 - \beta)d(Tx, Ty) \leq 2\beta d(x, Tx) + \beta d(x, y),$$

which yields

$$d(Tx,Ty) \le \frac{2\beta}{1-\beta} d(x,Tx) + \frac{\beta}{1-\beta} d(x,y). \tag{2}$$

If (z_3) holds, then similarly we get

$$d(Tx,Ty) \le \frac{2\gamma}{1-\gamma} d(x,Tx) + \frac{\gamma}{1-\gamma} d(x,y). \tag{3}$$

Therefore, denoting

$$\delta = \max\left\{\alpha,\, \frac{\beta}{1-\beta},\, \frac{\gamma}{1-\gamma}\right\},$$

we have $0 \le \delta < 1$ and then, for all $x, y \in X$, the following inequality

$$d(Tx,Ty) \le 2\delta\, d(x,Tx) + \delta\, d(x,y). \tag{4}$$

holds. In a similar manner we obtain

$$d(Tx,Ty) \le 2\delta\, d(x,Ty) + \delta\, d(x,y), \tag{5}$$

valid for all $x, y \in X$.

From (4) it follows that card $F_T \le 1$. We will show that T has a (unique) fixed point. Let $x_0 \in X$ be arbitrary and $\{x_n\}_{n=0}^{\infty}$,

$$x_n = T^n x_0, \qquad n = 0, 1, 2, \dots,$$

be the Picard iteration associated to T.

If $x := x_n,\ y := x_{n-1}$ are two successive approximations, then by (16) we have

$$d(x_{n+1}, x_n) \le \delta\, d(x_n, x_{n-1}).$$

From this we deduce that $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence, and hence a convergent sequence, too. Let $x^* \in X$ be its limit. In particular we have

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = 0.$$

By the triangle inequality and (4) we get

$$d(x^*, Tx^*) \le d(x^*, x_{n+1}) + d(Tx_n, Tx^*)$$

$$\le d(x^*, x_{n+1}) + \delta d(x^*, x_n) + 2\delta d(x_n, Tx_n),$$

which, by letting $n \to \infty$, yields

$$d(x^*, Tx^*) = 0 \Longleftrightarrow x^* = Tx^*,$$

since

$$d(x_n, Tx_n) = d(x_n, x_{n+1}) \to 0,$$

and therefore

$$F_T = \{x^*\} \quad \text{and} \quad x_n \to x^* \qquad (n \to \infty),$$

for each $x_0 \in X$.

Zamfirescu's theorem generalizes several classical fixed point results, including Banach's contraction principle, Kannan's theorem and Chatterjea's theorem. The theorem shows that it is sufficient for one of the three contractive conditions to hold for each pair of points. Under these assumptions, the mapping possesses a unique fixed point and the Picard iteration converges to this fixed point.

2.4 Maia's Fixed Point Theorem

Definition 2.4.1. Let (X,d) be a nonempty set. A map $T : X \mapsto X$ is said to be a *Bessaga mapping* if there exists $x^* \in X$ such that

$$F_T^n = x^* \text{ for all } n \in \mathbb{N}$$

Every Picard mapping is a Bessaga mapping . However , the converse is not necessarily true . Hence,the class of Bessaga mappings is broader than the class of Picard operators.

Theorem 2.4.1. *Let X be a nonempty set, $T : X \mapsto X$ a mapping satisfying $F_T^n = x^*$ for all $n \in \mathbb{N}$ and $a \in (0, 1)$ given number. Then there exists a metric d on X such that*

(a) (X,d) is a complete metric space;

(b) T is an a-contraction with respect to d.

Theorem 2.4.2. *Let X be a nonempty set, d and ρ two metrics on X and $T : X \mapsto X$ a mapping. Assume that*

(i) $d(x, y) \leq \rho(x, y)$, for all $x, y \in X$;

(ii) (X,d) is a complete metric space;

(iii) $T : (X, d) \mapsto (X, d)$ *is continuous;*

(iv) $T : (X, \rho) \mapsto (X, \rho)$ *is an* $a-$*contraction with* $a \in [0, 1)$.

Then T is a Picard mapping.

Proof : Let $x_0 \in X$ be arbitrary and let

$$\{x_n\}_{n=0}^{\infty}, \qquad x_n = T^n x_0, \quad n = 0, 1, 2, \ldots,$$

be the Picard iteration associated to T.

From (iv), using the same arguments as in the proof of Theorem 2.1.1, we deduce that $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence in (X, ρ).

By (i), it follows that $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence in (X, d) as well, and by (ii), we deduce that it converges to a certain x^* in X.

Now, by (iii), $x^* \in F_T$ and, by (iv),

$$F_T = \{x^*\}.$$

Remarks 2.4.1. Maia‘s theorem provides a unified framework for studying fixed points of generalized contractive mappings . It demonstrates that the convergence of Picard iteration can be guaranteed even when the mapping is not contractive with respect to the complete metric, thereby extending the applicability of Banach‘s contraction principle to a wider class of operators.

Chapter 3

The Krasnoselskij Iteration

3.1 Nonexpansive Operators and Krasnoselskij Iteration

Picard iteration is a powerful method for approximating fixed points of contractive mappings. However, for nonexpansive mappings, Picard iteration may fail to converge. To overcome this difficulty, Krasnoselskij introduced the iteration

$$x_{n+1} = (1-\lambda)x_n + \lambda T x_n, \qquad 0 < \lambda < 1,$$

which combines the current iterate with its image under the operator T. This averaging process improves convergence properties and is widely used in fixed point theory.
This chapter studies Krasnoselskij iteration for nonexpansive, strictly pseudocontractive, generalized pseudocontractive and pseudo φ-contractive operators.

Existence of Fixed Points

The foundation of the theory is provided by the Browder-Gohde-Kirk Fixed Point Theorem.

Theorem 3.1.1. *Let C be a closed, bounded and convex subset of a Hilbert space H, and* $T : C \mapsto C$ *be a nonexpansive operator . Then T possesses at least one fixed point.*

Proof : For a fixed element $v_0 \in C$ and a number s with $0 < s < 1$, we denote

$$U_s(x) = (1-s)v_0 + sTx, \qquad x \in C.$$

Since C is convex and closed, we deduce that

$$U_s : C \to C$$

is an s-contraction ,it has a unique fixed point, say u_s.
On the other hand, since C is closed, convex and bounded in the Hilbert space H, it is weakly compact. Hence we may find a sequence $\{s_j\}$ in $(0, 1)$ such that

$$s_j \to 1 \qquad (j \to \infty)$$

and

$$u_j = u_{s_j}$$

converges weakly to an element p of H.
Since C is weakly closed, p lies in C. We shall prove that p is a fixed point of T. If u is any arbitrary point in H, we have

$$\|u_j - u\|^2 = \|(u_j - p) + (p - u)\|^2 = \|u_j - p\|^2 + \|p - u\|^2 + 2\langle u_j - p, p - u\rangle,$$

where

$$2\langle u_j - p, p - u\rangle \to 0 \qquad (j \to \infty),$$

since $u_j - p$ converges weakly to zero in H.
Setting $u = Tp$ above, we obtain

$$\lim_{j\to\infty} \left(\|u_j - Tp\|^2 - \|u_j - p\|^2 \right) = \|p - Tp\|^2.$$

Moreover, since $s_j \to 1$ and $U_{s_j} u_j = u_j$, we have

$$\begin{aligned} Tu_j - u_j &= [s_j Tu_j + (1 - s_j)v_0] - u_j + (1 - s_j)(Tu_j - v_0) \\ &= (U_{s_j} u_j - u_j) + (1 - s_j)(Tu_j - v_0) \\ &= 0 + (1 - s_j)(Tu_j - v_0) \to 0, \end{aligned}$$

as $j \to \infty$, and therefore

$$\lim_{j\to\infty} \|Tu_j - u_j\| = 0.$$

On the other hand, since T is nonexpansive, we have

$$\|Tu_j - Tp\| \leq \|u_j - p\|,$$

and hence

$$\|u_j - Tp\| \leq \|u_j - Tu_j\| + \|Tu_j - Tp\| \leq \|u_j - Tu_j\| + \|u_j - p\|.$$

Thus

$$\limsup_{j\to\infty}\big(\|u_j - Tp\| - \|u_j - p\|\big) \leq \lim_{j\to\infty} \|u_j - Tu_j\| = 0.$$

Due to the boundedness of C, we have also

$$\begin{aligned}\limsup_{j\to\infty}\Big(\|u_j - Tp\|^2 - \|u_j - p\|^2\Big) &= \limsup_{j\to\infty}(\|u_j - Tp\| - \|u_j - p\|) \\ &\qquad \times (\|u_j - Tp\| + \|u_j - p\|) \\ &\leq 0.\end{aligned}$$

Therefore,

$$\lim_{j\to\infty}\Big(\|u_j - Tp\|^2 - \|u_j - p\|^2\Big) = 0,$$

and hence

$$\|p - Tp\|^2 = 0,$$

that is,

$$p = Tp.$$

Therefore p is a fixed point of T.

Definition 3.1.1. Let H be a Hilbert space and C a subset of H. A mapping$T : C \mapsto H$ is called *demicompact* if it has the property that whenever u_n is a bounded sequence in H and $Tu_n - u_n$ is strongly convergent, then there exists a subsequence $u_n k$ of u_n which is strongly convergent.

lemma 3.1.1. *Let C be a bounded closed convex subset of a Hilbert space H and $T : C \mapsto C$ be a nonexpansive and demicompact operator. Then the set F_T of fixed points of T is a nonempty convex set.*

Proof : Let $x, y \in F(T)$ and $\lambda \in [0, 1]$. Define

$$u_\lambda = (1 - \lambda)x + \lambda y.$$

Since T is nonexpansive,

$$\|Tu_\lambda - x\| = \|Tu_\lambda - Tx\| \leq \|u_\lambda - x\|,$$

and

$$\|Tu_\lambda - y\| = \|Tu_\lambda - Ty\| \leq \|u_\lambda - y\|.$$

Hence,

$$\|x - y\| \leq \|x - Tu_\lambda\| + \|Tu_\lambda - y\| \leq \|x - y\|.$$

Therefore, equality holds throughout. Consequently, there exist $a, b \in [0, 1]$ such that

$$x - Tu_\lambda = a(x - u_\lambda),$$

and

$$y - Tu_\lambda = b(y - u_\lambda).$$

Since

$$u_\lambda = (1 - \lambda)x + \lambda y,$$

it follows that

$$Tu_\lambda = u_\lambda.$$

Thus,

$$u_\lambda \in F(T),$$

which proves that $F(T)$ is convex.

Theorem 3.1.2. *Let C be a bounded closed convex subset of a Hilbert space H and $T : C \mapsto C$ be a nonexpansive and demicompact operator .Then the set F_T of fixed points of T is a nonempty convex set and for any given x_0 in C and any fixed number λ with $0 \leq \lambda \leq 1$, the Krasnoselkij iteration $\{x_n\}_{n=0}^{\infty}$ given by*

$$x_n + 1 = (1{-}\lambda)x_n + \lambda x_n, n = 0, 1, 2, ...$$

converges (strongly) to a fixed point of T.

Proof : The first part follows by Lemma 3.1.1 For any $x_0 \in$,the sequence $\{x_n\}_{n=0}^{\infty}$,$n = 0$ given by($x_n + 1 = (1{-}\lambda)x_n + \lambda x_n$,$n = 0, 1, 2, ...$)lies in C and is bounded.Let p be a fixed point of T,and,so of the averaged map λ . We first prove that the sequence $\{x_n - Tx_n\}_{n\in\mathbb{N}}$ converges strongly to zero.

Indeed,

$$x_{n+1} - p = (1-\lambda)x_n + \lambda Tx_n - p = (1-\lambda)(x_n - p) + \lambda(Tx_n - p).$$

On the other hand, for any constant a,

$$a(x_n - Tx_n) = a(x_n - p) - a(Tx_n - p).$$

Then

$$\|x_{n+1} - p\|^2 = (1-\lambda)^2\|x_n - p\|^2 + \lambda^2\|Tx_n - p\|^2 + 2\lambda(1-\lambda)\langle Tx_n - p, x_n - p\rangle,$$

and

$$a^2\|x_n - Tx_n\|^2 = a^2\|x_n - p\|^2 + a^2\|Tx_n - p\|^2 - 2a^2\langle Tx_n - p, x_n - p\rangle.$$

Hence, summing up the corresponding sides of the two inequalities and using the fact that T is nonexpansive and $Tp = p$, we get

$$\begin{aligned}\|x_{n+1} - p\|^2 + a^2\|x_n - Tx_n\|^2 \leq \left[2a^2 + \lambda^2 + (1-\lambda)^2\right]\|x_n - p\|^2 \\ + \left[2\lambda(1-\lambda) - 2a^2\right]\langle Tx_n - p, x_n - p\rangle.\end{aligned}$$

If we choose now an a such that

$$a^2 \leq \lambda(1-\lambda),$$

then from the last inequality we obtain

$$\begin{aligned}\|x_{n+1} - p\|^2 + a^2\|x_n - Tx_n\|^2 &\leq \left(2a^2 + \lambda^2 + (1-\lambda)^2 + 2\lambda(1-\lambda) - 2a^2\right)\|x_n - p\|^2 \\ &= \|x_n - p\|^2,\end{aligned}$$

where we used the Cauchy–Schwarz inequality,

$$\langle Tx_n - p, x_n - p\rangle \leq \|Tx_n - p\| \, \|x_n - p\| \leq \|x_n - p\|^2.$$

Letting now

$$a^2 = \lambda(1-\lambda) > 0,$$

and summing up the obtained inequality, we get

$$a^2\|x_n - Tx_n\|^2 \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2.$$

Summing for $n = 0$ to $n = N$,

$$\lambda(1-\lambda)\sum_{n=0}^{N}\|x_n - Tx_n\|^2 \leq \sum_{n=0}^{N}\Big(\|x_n - p\|^2 - \|x_{n+1} - p\|^2\Big),$$

hence

$$\lambda(1-\lambda)\sum_{n=0}^{N}\|x_n - Tx_n\|^2 \leq \|x_0 - p\|^2 - \|x_{N+1} - p\|^2 \leq \|x_0 - p\|^2.$$

Therefore,

$$\sum_{n=0}^{\infty}\|x_n - Tx_n\|^2 < \infty,$$

which shows that

$$\|x_n - Tx_n\| \to 0, \qquad n \to \infty.$$

As T is demicompact,it results that there exists a strongly convergent subsequence x_{n_i} such that $x_{n_i} \mapsto p \in F_T$. Since T is nonexpansive , $T_{x_{n_i}} \mapsto T_p$ and $Tp = p$ .The convergence of the entire sequence $\{x_n\}_{n=0}^{\infty}$ to p now follows from the inequality $\|x_n + 1 - p\| \leq \|x_n - p\|$ which can be deduced from the nonexpansiveness of T and is valid for each n.

Remarks 3.1.1. This theorem establishes that the averaged iteration successfully approximates a fixed point whenever the operator is both nonexpansive and demicomicompact.

Theorem 3.1.3. *Let C be a bounded closed convex subset of a Hilbert space and let $T : C \mapsto C$ be a nonexpansive operator. Then, for every $x_0 \in C$ the Krasnoselskij iteration converges weakly to a fixed point of T.*

Proof : Let F_T be the set of all fixed points of T in C (which is nonempty, by Theorem 3.1.1 and convex, by Lemma 3.1.1). As T is nonexpansive, for each $p \in F_T$ and each n we have

$$\|x_{n+1} - p\| \leq \|x_n - p\|,$$

which shows that the function

$$g(p) = \lim_{n\to\infty} \|x_n - p\|$$

is well defined and is a lower semicontinuous convex function on F_T. Let

$$d_0 = \inf\{g(p) : p \in F_T\}.$$

For each $\varepsilon > 0$, the set

$$F_\varepsilon = \{y : g(y) \leq d_0 + \varepsilon\}$$

is closed, convex, nonempty and bounded and, hence, weakly compact. Therefore

$$\bigcap_{\varepsilon>0} F_\varepsilon \neq \emptyset,$$

and in fact

$$\bigcap_{\varepsilon>0} F_\varepsilon = \{y : g(y) = d_0\} \equiv F_0.$$

Moreover, F_0 contains exactly one point. Indeed, since F_0 is convex and closed, for $p_0, p_1 \in F_0$, and

$$p_\lambda = (1 - \lambda)p_0 + \lambda p_1,$$

we have

$$
\begin{aligned}
g^2(p_\lambda) &= \lim_{n\to\infty} \|p_\lambda - x_n\|^2 \\
&= \lim_{n\to\infty} \|\lambda(p_1 - x_n) + (1-\lambda)(p_0 - x_n)\|^2 \\
&= \lim_{n\to\infty} \Big(\lambda^2\|p_1 - x_n\|^2 + (1-\lambda)^2\|p_0 - x_n\|^2 \\
&\qquad\quad + 2\lambda(1-\lambda)\langle p_1 - x_n,\ p_0 - x_n\rangle\Big) \\
&= \lim_{n\to\infty} \Big(\lambda^2\|p_1 - x_n\|^2 + (1-\lambda)^2\|p_0 - x_n\|^2 \\
&\qquad\quad + 2\lambda(1-\lambda)\|p_1 - x_n\|\,\|p_0 - x_n\|\Big) \\
&\quad + \lim_{n\to\infty} \Big\{2\lambda(1-\lambda)\big[\langle p_1 - x_n,\ p_0 - x_n\rangle \\
&\qquad\quad - \|p_1 - x_n\|\,\|p_0 - x_n\|\big]\Big\} \\
&= g^2(p) + \lim_{n\to\infty} \Big\{2\lambda(1-\lambda)\big[\langle p_1 - x_n,\ p_0 - x_n\rangle \\
&\qquad\quad - \|p_1 - x_n\|\,\|p_0 - x_n\|\big]\Big\}.
\end{aligned}
$$

Hence

$$\lim_{n\to\infty} \Big\{2\lambda(1-\lambda)\big[\langle p_1 - x_n,\ p_0 - x_n\rangle - \|p_1 - x_n\|\,\|p_0 - x_n\|\big]\Big\} = 0.$$

Since

$$\|p_1 - x_n\| \to d_0 \qquad \text{and} \qquad \|p_0 - x_n\| \to d_0,$$

the latter relation implies that

$$\|p_1 - p_0\|^2 = \|(p_1 - x_n) + (x_n - p_0)\|^2$$

$$= \|p_1 - x_n\|^2 + \|x_n - p_0\|^2 - 2\langle p_1 - x_n,\ p_0 - x_n\rangle$$

$$\to d_0^2 + d_0^2 - 2d_0^2 = 0,$$

giving a contradiction.

Now, in order to show that

$$x_n = U_\lambda^n x_0 \rightharpoonup p_0,$$

it suffices to assume that

$$x_{n_j} \rightharpoonup p$$

for an infinite subsequence and then prove that $p = p_0$.
$p \in F(T)$.
Considering the definition of g and the fact that

$$x_{n_j} \rightharpoonup p,$$

we have

$$\|x_{n_j} - p_0\|^2 = \|x_{n_j} - p + p - p_0\|^2$$

$$= \|x_{n_j} - p\|^2 + \|p - p_0\|^2 - 2\langle x_{n_j} - p,\ p - p_0\rangle$$

$$\to g^2(p) + \|p - p_0\|^2 = g^2(p_0) = d_0^2.$$

Since

$$g^2(p) \geq d_0^2,$$

the last equality implies that

$$\|p - p_0\|^2 \leq 0,$$

which means that

$$p = p_0.$$

Remarks 3.1.2. Even without demicompactness , convergence to a fixed point is guaranteed in the weak sense.

3.2 Strictly Pseudocontractive Operators

In this section, we study the convergence of the Krasnoselskij iteration for the class of strictly pseudocontractive operators. This class is broader than the class of nonexpansive mappings and therefore extends the applicability of fixed point approximation methods.

Definition 3.2.1. Let C be a subset of a Hilbert space H. A mapping

$$T : C \to C$$

is said to be ***strictly pseudocontractive*** if there exists a constant $k < 1$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2,$$

for all $x, y \in C$, where I denotes the identity operator.

Strictly pseudocontractive mappings form a larger class than nonexpansive mappings and arise naturally in several problems of nonlinear analysis.

Theorem 3.2.1. *Let C be a bounded closed convex subset of a Hilbert space H, and let*

$$T : C \to C$$

be a strictly pseudocontractive operator. Suppose there exists a constant $k < 1$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2, \qquad x, y \in C.$$

Then, for any $x_0 \in C$ and any fixed μ satisfying

$$0 < \mu < 1 - k,$$

the Krasnoselskij iteration

$$x_{n+1} = (1 - \mu)x_n + \mu T x_n, \qquad n = 0, 1, 2, \ldots,$$

converges weakly to a fixed point p of T. Furthermore, if T is demicompact, then the convergence is strong.

Proof : We denote as usual

$$T_t = (1-t)I + tT$$

and show that T_t is nonexpansive. Indeed, by the pseudocontractiveness condition , it follows that

$$U = I - T$$

is strongly monotone, i.e.,

$$\langle Ux - Uy,\ x - y\rangle \geq m\|Ux - Uy\|^2, \qquad m = \frac{1-k}{2} > 0.$$

Then, for any $t > 0$,

$$\|T_t x - T_t y\|^2 = \|(I - tU)x - (I - tU)y\|^2$$

$$= \|x - y\|^2 + t^2\|Ux - Uy\|^2 - 2t\langle Ux - Uy,\ x - y\rangle$$

$$\leq \|x - y\|^2 + \big(t^2 - 2tm\big)\|Ux - Uy\|^2.$$

Now, if we take

$$t \leq 2m = 1 - k,$$

then from the preceding inequality we obtain

$$\|T_t x - T_t y\| \leq \|x - y\|, \qquad x, y \in C,$$

which shows that T_t is nonexpansive.
T_t (and therefore T) has a fixed point p_0 in C, and for any fixed λ with

$$0 < \lambda < 1,$$

the Krasnosel'skij iteration

$$x_n = (T_t)_\lambda^n(x_0)$$

associated to T_t converges weakly to some fixed point p of T in C.

But the iteration function $(T_t)_\lambda$ is in fact

$$(T_t)_\lambda = (1-\lambda)I + \lambda T_t$$

$$= (1-\lambda)I + \lambda\big[(1-t)I + tT\big]$$

$$= (1-\lambda t)I + \lambda t T = T_\mu,$$

with

$$\mu = \lambda t < t \leq 1-k.$$

In order to prove the second part of the theorem, it is enough to show that T_μ is demicompact.

But this follows from the demicompactness of T using the equality

$$T_\mu x - x = \mu(Tx - x),$$

valid for every $x \in C$.

The significance of the above theorem lies in the fact that the convergence of Krasnoselskij iteration is no longer restricted to nonexpansive operators. By introducing the class of strictly pseudocontractive mappings, the theorem considerably enlarges the range of operators for which fixed points can be approximated. The theorem also highlights the importance of demicompactness. Without this assumption, only weak convergence is guaranteed. When demicompactness is present, weak convergence is strengthened to strong convergence.

3.3 Lipschitzian and Generalized Pseudocontractive Operators

In the previous section, Krasnoselskij iteration was studied for strictly pseudocontractive operators. We now consider a broader class of mappings known as generalized pseudocontractive operators. When such operators are also Lipschitzian, strong convergence results for the Krasnoselskij iteration can be established.
Although strictly pseudocontractive operators and generalized pseudocontractive operators are closely related, neither class is contained in the other. Consequently, it is important to investigate generalized pseudocontractive mappings independently.

Definition 3.3.1. Let H be a Hilbert space and let

$$T : H \to H.$$

The operator T is called a *generalized pseudocontractive operator* if there exists a constant $r > 0$ such that

$$\|Tx - Ty\|^2 \le r^2\|x - y\|^2 + \|Tx - Ty - r(x - y)\|^2,$$

for all $x, y \in H$.

The above condition is equivalent to

$$\langle Tx - Ty, x - y\rangle \le r\|x - y\|^2,$$

for all $x, y \in H$.
Furthermore,

$$\langle (I - T)x - (I - T)y, x - y\rangle \ge (1 - r)\|x - y\|^2,$$

which shows that the operator $I - T$ is strongly monotone whenever $r < 1$.

Definition 3.3.2. An operator $T : H \to H$ is said to be *Lipschitzian* if there exists a constant $s > 0$ such that

$$\|Tx - Ty\| \le s\|x - y\|, \qquad x, y \in H.$$

The constant s is called the *Lipschitz constant* of T.

The combination of generalized pseudocontractivity and Lipschitz continuity leads to an important convergence theorem for Krasnoselskij iteration.

Theorem 3.3.1. *Let K be a nonempty closed convex subset of a real Hilbert space and let*

$$T : K \to K$$

be a generalized pseudocontractive and Lipschitzian operator with constants r and s satisfying

$$0 < r < 1, \qquad r \le s.$$

Then:

1. *T has a unique fixed point p.*
2. *For every $x_0 \in K$, the Krasnoselskij iteration*

   $$x_{n+1} = (1 - \lambda)x_n + \lambda T x_n, \qquad n = 0, 1, 2, \ldots,$$

   converges strongly to p whenever

   $$0 < \lambda < \frac{2(1 - r)}{1 - 2r + s^2}.$$

3. *Both the* a priori *estimate*

   $$\|x_n - p\| \le \tfrac{\theta^n}{1-\theta} \|x_1 - x_0\|, \qquad n = 1, 2, \ldots$$

   and a posteriori *estimate*

   $$\|x_n - p\| \le \tfrac{\theta}{1-\theta} \|x_n - x_{n-1}\|, \qquad n = 1, 2, \ldots$$

hold, with

$$\theta = \Big((1-\lambda)^2 + 2\lambda(1-\lambda)r + \lambda^2 s^2\Big)^{1/2}.$$

Proof : We consider the averaged operator F associated to T,

$$F_\lambda x = (1-\lambda)x + \lambda T x, \qquad x \in K.$$

for all $\lambda \in [0,1]$.
Since K is convex, we have that

$$F(K) \subset K$$

for each $\lambda \in [0,1]$.
As a closed subset of a Hilbert space, K is a complete metric space. We claim that F is a θ-contraction with θ given by $\theta = \Big((1-\lambda)^2 + 2\lambda(1-\lambda)r + \lambda^2 s^2\Big)^{1/2}$.
Indeed, since T is generalized pseudo-contractive and Lipschitzian, we have

$$\begin{aligned}
\|Fx - Fy\|^2 &= \|(1-\lambda)x + \lambda Tx - (1-\lambda)y - \lambda Ty\|^2 \\
&= \|(1-\lambda)(x-y) + \lambda(Tx - Ty)\|^2 \\
&= (1-\lambda)^2\|x-y\|^2 \\
&\quad + 2\lambda(1-\lambda)\langle Tx - Ty,\ x - y\rangle \\
&\quad + \lambda^2\|Tx - Ty\|^2 \\
&\le \Big((1-\lambda)^2 + 2\lambda(1-\lambda)r + \lambda^2 s^2\Big)\|x-y\|^2.
\end{aligned}$$

Therefore,

$$\|Fx - Fy\| \le \theta\,\|x-y\|, \qquad \forall\, x, y \in K.$$

Hence F is a θ-contraction.
The theorem establishes that a generalized pseudocontractive and Lipschitzian operator defined on a closed convex subset of a Hilbert space possesses a unique fixed point.

Moreover, it guarantees that the Krasnoselskij iteration converges strongly to this fixed point for every initial approximation, provided that the relaxation parameter λ satisfies the prescribed condition. The theorem therefore provides both the existence of a solution and a reliable iterative procedure for approximating it.

Theorem 3.3.2. *Under the assumptions of Theorem 3.3.1, the fastest Krasnoselskij iteration in the family*

$$x_{n+1} = (1-\lambda)x_n + \lambda T x_n$$

is obtained when

$$\lambda_{\text{min}} = \frac{1-r}{1-2r+s^2}.$$

Proof :

$$f(x) = (1-x)^2 + 2x(1-x)r + x^2 s^2,$$

Expanding,

$$f(x) = (1-2r+s^2)x^2 - 2(1-r)x + 1, \qquad x \in (0,a),$$

where

$$a = \frac{2(1-r)}{1-2r+s^2}.$$

the Lipschitzian constant s and the generalized pseudo-contractivity con- stant r fulfill the conditions $0 \leq r \leq 1$and $r \leq s$.

$$1-2r+s^2 \geq (1-r)^2 > 0.$$

Hence f admits a minimum, attained at

$$x = \lambda_{\text{min}},$$

where

$$\lambda_{\text{min}} = \frac{1-r}{1-2r+s^2}.$$

The minimum value of f is

$$f_{\text{min}} = \frac{s^2 - r^2}{1 - 2r + s^2}.$$

Therefore, the minimum value of θ is

$$\theta_{\text{min}} = \left(\frac{s^2 - r^2}{1 - 2r + s^2} \right)^{1/2}.$$

that completes the proof.

This theorem identifies the optimal choice of the relaxation parameter λ for the Krasnoselskij iteration. Among all admissible values of λ , the value

$$\lambda_{\text{min}} = \frac{1 - r}{1 - 2r + s^2}.$$

produces the fastest convergence to the fixed point. Thus, while Theorem 3.3.1guarantees convergence, Theorem 3.3.2 determines the most efficient iteration within the Krasnoselskij family. This result is particularly important in practical computations, where the speed of convergence plays a crucial role in reducing computational effort.

3.4 Pseudo φ-Contractive Operators

Pseudo φ-contractive operators provide a unified framework for studying several important classes of nonlinear operators arising in fixed point theory. This class includes contractions, nonexpansive operators, pseudocontractive operators, strictly pseudocontractive operators and generalized pseudocontractive operators as special cases. Hence, many convergence results can be investigated within a single theoretical setting.

The relationships among these classes can be expressed as

$$C_0 \subset C_1 \subset C_2 \subset C_3 \subset C_4,$$

where C_0 denotes the class of contractions, C_1 the class of nonexpansive operators, C_2 the class of strictly pseudocontractive operators, C_3 the class of pseudocontractive operators and C_4 the class of generalized pseudocontractive operators.

A pseudo φ-contractive operator is defined by means of a comparison function φ, which generalizes the contraction condition and allows a wider class of mappings to be studied.

Definition 3.4.1. Let H be a Hilbert space and let $T : H \to H.$ Then T is said to be a *(strictly) pseudo φ-contractive operator* if there exist constants $a, b, c \in \mathbb{R}$ satisfying

$$a + b + c = 1,$$

and a comparison function

$$\varphi : \mathbb{R}_+ \to \mathbb{R}_+,$$

such that

$$a\|x - y\|^2 + b\langle Tx - Ty, x - y\rangle + c\|Tx - Ty\|^2 \leq \varphi^2(\|x - y\|),$$

for all $x, y \in H.$
If the inequality is strict whenever $x \neq y$, then T is called a strictly pseudo φ-contractive operator.

Examples

The class of pseudo φ-contractive operators contains several important classes of operators as special cases.

1. Every Lipschitzian operator is a pseudo φ-contractive operator for a suitable comparison function φ.

2. Every pseudocontractive operator is a pseudo φ-contractive operator.

3. Every generalized pseudocontractive operator is a pseudo φ-contractive operator.

4. Every strictly pseudocontractive operator is a pseudo φ-contractive operator.

5. Every strongly pseudocontractive operator is also a pseudo φ-contractive operator.

These examples show that the class of pseudo φ-contractive operators unifies several important operator classes studied in fixed point theory.

Theorem 3.4.1. *Let K be a nonempty closed convex subset of a real Hilbert space H and let $T : K \to K$ be a strictly φ-contractive operator. Then:*

1. *T has a unique fixed point $p \in K$.*
2. *For each $x_0 \in K$, the Krasnoselskij iteration*

$$x_{n+1} = (1-\lambda)x_n + \lambda T x_n, \qquad n = 0, 1, 2, \dots,$$

converges strongly to p, for every $\lambda \in (0, 1)$.

3. *If, additionally, φ is a (c)-comparison function, then*

$$\|x_n - p\| \leq s(\|x_n - x_{n+1}\|), \qquad n = 1, 2, \dots,$$

where

$$s(t) = \sum_{k=0}^{\infty} \varphi^k(t)$$

denotes the sum of the comparison series.

Proof : Let

$$F(x) = (1-\lambda)x + \lambda T x, \qquad x \in K,$$

where $\lambda \in (0, 1)$.
Since K is convex and $T(K) \subseteq K$, it follows that $F(K) \subseteq K$.
Moreover, for every $x, y \in K$,

$$\|F(x) - F(y)\|^2 \leq \varphi^2(\|x - y\|).$$

Hence,

$$\|F(x) - F(y)\| \leq \varphi(\|x - y\|),$$

which shows that F is a φ-contractive mapping.
Since K is a closed subset of the Hilbert space H, it is complete. Therefore, every φ-contractive mapping on a complete metric space has a unique fixed point, and the corresponding Picard iteration converges to this fixed point. Hence, F has a unique fixed point $p \in K$.
Since

$$F(p) = p \iff (1-\lambda)p + \lambda T p = p,$$

it follows that $Tp = p$. Thus, p is the unique fixed point of T.
Further, the Picard iteration associated with F is

$$x_{n+1} = F(x_n) = (1 - \lambda)x_n + \lambda T x_n,$$

which is precisely the Krasnoselskij iteration. Therefore,

$$x_n \to p \qquad \text{as } n \to \infty.$$

Finally, if φ is a (c)-comparison function, then the following a posteriori estimate holds:

$$\|x_n - p\| \le s(\|x_n - x_{n+1}\|), \qquad n = 1, 2, \ldots,$$

where

$$s(t) = \sum_{k=0}^{\infty} \varphi^k(t).$$

This completes the proof.

3.5 Quasi-Nonexpansive Operators

Quasi nonexpansive operators form an important class of mappings in fixed point theory. Every nonexpansive operator is quasi nonexpansive, but the converse is not necessarily true. Consequently, convergence results for quasi nonexpansive operators extend the applicability of iterative methods to a broader class of mappings.

Definition 3.5.1. A mapping $T : D \to D$ is said to be *quasi nonexpansive* if

$$\|Tx - p\| \le \|x - p\|, \qquad x \in D,$$

for every fixed point $p \in F(T)$, where

$$F(T) = \{x \in D : Tx = x\}$$

denotes the set of fixed points of T.
The convergence of Krasnoselskij iteration for quasi nonexpansive operators can be established under an additional contractive-type assumption known as the Frum–Ketkov condition.

Frum–Ketkov Condition

An operator $T : D \to D$ is said to satisfy the Frum–Ketkov condition if there exist a compact convex subset K of D and a constant $k \in (0, 1)$ such that

$$d(Tx, K) \leq k\, d(x, K), \qquad x \in D,$$

where

$$d(x, K) = \inf\{\|x - y\| : y \in K\}$$

denotes the distance from the point x to the set K.

This condition implies that the image Tx is always closer to the compact set K than the point x itself. Therefore, repeated application of the operator gradually moves the iterates toward the set K, which plays an important role in establishing convergence. The main convergence result of this section is given below.

Theorem 3.5.1. *Let D be a closed convex subset of a strictly convex Banach space and let $T : D \to D$ be a conditionally quasi nonexpansive operator satisfying the Frum–Ketkov condition. Then, for every $x_0 \in D$ and every $\lambda \in (0, 1)$, the Krasnoselskij iteration*

$$x_{n+1} = (1 - \lambda)x_n + \lambda T x_n$$

converges to a fixed point of T.

Proof : Consider the averaged operator

$$T_\mu = (1 - \mu)I + \mu T,$$

where $0 < \mu < 1 - k$. Using the strict pseudocontractivity condition

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2,$$

it can be shown that T_μ is nonexpansive. Hence, the Krasnoselskij iteration

$$x_{n+1} = T_\mu x_n = (1 - \mu)x_n + \mu T x_n$$

is well defined.
Since C is a bounded closed convex subset of a Hilbert space, the convergence results for nonexpansive mappings imply that the sequence $\{x_n\}$ converges weakly to a fixed point $p \in FT$.
If T is additionally demicompact, then weak convergence together with the demicompactness property yields strong convergence. Therefore,

$$x_n \to p,$$

where p is a fixed point of T.

Remarks 3.5.1. This theorem guarantees the convergence of the Krasnoselskij iteration for conditionally quasi nonexpansive operators satisfying the Frum–Ketkov condition. The theorem shows that, starting from any initial approximation, the iterative sequence converges to a fixed point of the operator. Thus, the result extends the convergence theory of Krasnoselskij iteration beyond nonexpansive mappings and demonstrates its effectiveness for a wider class of nonlinear operators in strictly convex Banach spaces.

BIBLIOGRAPHY

1. V. Berinde, *Iterative Approximation of Fixed Points*, Lecture Notes in Mathematics, Vol. 1912, Springer, Berlin–Heidelberg, 2007.

2. W. A. Kirk and B. Sims (Eds.), *Handbook of Metric Fixed Point Theory*, Kluwer Academic Publishers, Dordrecht, 2001.

3. D. R. Sahu, *Topological Fixed Point Theory and Its Applications*, Springer, Cham, 2022.

4. Ravi P. Agarwal, Maria Meehan, and Donal O'Regan, *Fixed Point Theory and Applications*, Cambridge University Press, 2001.

5. K. Goebel and W. A. Kirk, *Topics in Metric Fixed Point Theory*, Cambridge University Press, 1990.