INDUCED PATH CONVEXITY

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Introduction

Graph theory is a fundamental branch of discrete mathematics that studies structures consisting of vertices (or nodes) and edges connecting pairs of vertices. Since its inception, graph theory has developed into a rich and active area of mathematical research with numerous applications in computer science, communication networks, transportation systems, biology, social sciences, operations research, electrical engineering, and artificial intelligence. Graphs provide a natural mathematical framework for modeling relationships among objects, making them indispensable in both theoretical studies and practical applications.

One of the important areas of graph theory is graph convexity, which extends the classical notion of convexity from Euclidean geometry to discrete structures. In Euclidean space, a set is said to be convex if it contains the entire line segment joining every pair of its points. Convexity plays a central role in mathematics, particularly in optimization theory, where it provides important structural properties that simplify the search for optimal solutions. The abstract formulation of convexity has enabled its extension beyond geometry to various mathematical structures, including graphs.

In graph theory, the notion of convexity is obtained by replacing line segments with suitable classes of paths. A graph convexity is defined by a family of vertex subsets, called convex sets, that is closed under intersection and contains both the empty set and the entire vertex set. Depending on the type of paths considered, several notions of graph convexity have been introduced, each possessing its own structural and combinatorial properties.

Among the various graph convexities, geodesic convexity (or shortest path convexity) and induced-path convexity (also known as monophonic convexity) are two of the most widely studied. In geodesic convexity, convexity is defined using shortest paths between vertices. In contrast, monophonic convexity is based on induced paths (or chordless paths), that is, paths containing no edges between non-consecutive vertices. A set of vertices is said to be monophonically convex if it contains every vertex lying on every induced path joining any two of its vertices. Because induced paths preserve important structural properties of graphs, monophonic convexity has become an active topic of research in graph theory.

Associated with every graph convexity are several numerical parameters, known as convexity invariants, which describe different structural aspects of convex sets and convex hulls. Among the most important of these invariants are the hull number, Carathéodory number and Helly number. The hull number is the minimum cardinality of a vertex set whose convex hull is the entire vertex set of the graph. The Carathéodory number measures the minimum number of vertices required to generate a vertex in the convex hull of a set. The Helly number describes the intersection properties of families of convex sets. These parameters have been extensively studied under different graph convexities and provide valuable information about the underlying structure of graphs.

Over the past several decades, numerous researchers have investigated monophonic convexity and its associated convexity invariants for various classes of graphs. Their work has led to many important definitions, characterizations, algorithms, and theoretical results concerning hull sets, convex hulls, Carathéodory sets and Helly properties. These results have significantly contributed to the development of graph convexity and continue to motivate further research in the area.

The primary purpose of this dissertation is to present a comprehensive review of induced-path (monophonic) convexity and the classical convexity invariants associated with it. The dissertation brings together the fundamental concepts, definitions, and important known results available in the literature concerning the hull number, Carathéodory number and Helly number in monophonic convexity. Rather than presenting new theoretical results, this work aims to provide a clear and systematic exposition of the existing theory, making it accessible to readers who wish to understand the subject and its development.

The dissertation begins with the necessary preliminaries from graph theory and graph convexity. It then introduces induced-path (monophonic) convexity and discusses its basic properties. The subsequent chapters review the literature on the hull number, Carathéodory number and Helly number under monophonic convexity, highlighting important definitions, fundamental theorems, and known results for various classes of graphs.

Chapter 1

Preliminaries

This chapter presents the basic concepts and terminology required throughout the dissertation. We begin with fundamental definitions and results from graph theory, followed by the basic notions of graph convexity and convexity spaces. Finally, we introduce induced path (monophonic) convexity together with several concepts that will be used in the later chapters. Unless otherwise stated, all graphs considered in this dissertation are finite, simple, undirected, and connected.

1.1 Basic Concepts in Graph Theory

In this section, we recall some standard definitions and notation from graph theory that will be used throughout the dissertation.

Definition 1.1.1. *A* ***graph*** *is an ordered pair $G = (V, E)$, where V is a finite non-empty set of objects called* vertices *and E is a set of unordered pairs of distinct vertices, called* edges*. The sets V and E are called the vertex set and edge set of G, respectively.*

Definition 1.1.2. *The* ***order*** *of a graph G is the number of vertices of G, denoted by $|V(G)|$ or n, and the* ***size*** *of G is the number of edges of G, denoted by $|E(G)|$ or m.*

Definition 1.1.3. *Let u and v (not necessarily distinct) be vertices of a graph G. A u–v **walk** in G is a finite alternating sequence $u = u_0,\ e_1,\ u_1,\ e_2,\ \dots,\ e_k,\ u_k = v$ of vertices and edges, beginning with the vertex u and ending with the vertex v, such that $e_i = u_{i-1}u_i, i = 1, 2, \dots, k$. The number k is called the **length** of the walk. The walk is said to be **closed** if $u = v$, and **open** otherwise.A walk $u_0,\ e_1,\ u_1,\ e_2,\ u_2,\ \dots,\ e_k,\ u_k$ is completely determined by the sequence of its vertices $u_0,\ u_1,\ u_2,\ \dots,\ u_k$, and is therefore denoted simply by $u_0,\ u_1,\ u_2,\ \dots,\ u_k$. A walk in which all the vertices are distinct is called a **path**. A closed walk $u_0,\ u_1,\ u_2,\ \dots,\ u_k$ in which the vertices $u_0,\ u_1,\ u_2,\ \dots,\ u_{k-1}$ are distinct is called a **cycle**.*

Definition 1.1.4. *Two graphs G_1 and G_2 are said to be **isomorphic** if there exists a bijection $f : V(G_1) \to V(G_2)$ such that*

$$uv \in E(G_1) \iff f(u)f(v) \in E(G_2).$$

If G_1 and G_2 are isomorphic, we write $G_1 \cong G_2$.

Definition 1.1.5. *A graph H is a **subgraph** of a graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. If $V(H) = V(G)$, then H is called a **spanning subgraph** of G.*

*For a subset $S \subseteq V(G)$, the **induced subgraph** of G on S, denoted by $G[S]$, is the subgraph whose vertex set is S and whose edge set consists of all edges of G having both endpoints in S.*

Definition 1.1.6. *The **degree** of a vertex v in a graph G, denoted by $\deg(v)$, is the number of edges incident with v. The maximum and minimum degrees of G are denoted by $\Delta(G)$ and $\delta(G)$, respectively. A vertex of degree 0 is called an **isolated vertex**, while a vertex of degree 1 is called a **pendant vertex**.*

Definition 1.1.7. *A graph is called r**-regular** if every vertex has degree r. A 3-regular graph is called a **cubic graph**.*

Definition 1.1.8. *A graph is called **complete** if every pair of distinct vertices is adjacent. The complete graph on n vertices is denoted by K_n.*

Definition 1.1.9. *A graph G is called* ***bipartite*** *if its vertex set can be partitioned into two disjoint sets V_1 and V_2 such that every edge joins a vertex of V_1 to a vertex of V_2. If every vertex of V_1 is adjacent to every vertex of V_2, then G is called a* ***complete bipartite graph*** *and is denoted by $K_{r,s}$, where $|V_1| = r$ and $|V_2| = s$.*

Definition 1.1.10. *A* ***clique*** *in a graph is a set of pairwise adjacent vertices. The maximum cardinality of a clique in G is called the* ***clique number*** *of G and is denoted by $\omega(G)$.*

Definition 1.1.11. *Let $G = (V(G), E(G))$ be a connected graph, and let $u, v \in V(G)$. The* ***distance*** *between u and v, denoted by $d(u, v)$, is the length of a shortest u–v path in G. Such a path is called a* ***geodesic***. *The* ***diameter*** *of G, denoted by* $\operatorname{diam}(G)$*, is the maximum distance between any two vertices of G, that is,*

$$\operatorname{diam}(G) = \max\{d(u, v) : u, v \in V(G)\}.$$

Definition 1.1.12. *Let $G = (V(G), E(G))$ be a graph and let $u, v \in V(G)$. A u–v path*

$$P = u = v_0, v_1, \ldots, v_k = v$$

is called an ***induced path*** *(or* ***chordless path****) if the subgraph of G induced by the vertex set*

$$\{v_0, v_1, \ldots, v_k\}$$

contains exactly the edges of the path P. Equivalently, no two non-consecutive vertices of P are adjacent.

Definition 1.1.13. *Let $G = (V(G), E(G))$ be a graph. For a vertex $v \in V(G)$, the* ***open neighbourhood*** *of v is*

$$N(v) = \{u \in V(G) : uv \in E(G)\},$$

For a subset $S \subseteq V(G)$, the neighbourhood of S is defined by

$$N(S) = \bigcup_{v \in S} N(v).$$

Definition 1.1.14. *A vertex v is called **simplicial** if the subgraph induced by its neighborhood $N(v)$ is a complete graph. Every monophonic set or m-hull set of a graph G must contain all of its simplicial vertices. This implies that every graph containing a simplicial vertex satisfies $c_m(G) = n - 1$*

1.2 Graph Convexity

Graph convexity extends the classical notion of convexity from Euclidean spaces to graphs by replacing line segments with suitable classes of paths.

Definition 1.2.1. *For vertices $u, v \in V(G)$, the **geodesic interval** between u and v is*

$$I[u, v] = \{x \in V(G) : x \text{ lies on a } u\text{–}v \text{ geodesic}\}.$$

For a set $S \subseteq V(G)$,

$$I[S] = \bigcup_{u,v \in S} I[u, v].$$

Definition 1.2.2. *A set $S \subseteq V(G)$ is called **geodesically convex** if*

$$I[S] = S.$$

1.3 Convexity Spaces

The abstract theory of convexity spaces provides a unified framework for studying different convexities.

Definition 1.3.1. *A **convexity space** is a pair $(V, \mathcal{C})$, where $\mathcal{C}$ is a family of subsets of V satisfying*

1. *$\emptyset, V \in \mathcal{C}$;*

2. *arbitrary intersections of members of $\mathcal{C}$ belong to $\mathcal{C}$;*

3. unions of nested families of convex sets belong to $\mathcal{C}$.

Definition 1.3.2. *For a subset $S \subseteq V$, the smallest convex set containing S is called the* ***convex hull*** *of S and is denoted by $\langle S \rangle$*

Definition 1.3.3. *The* ***hull number*** *of a convexity space is the minimum cardinality of a set whose convex hull is the entire vertex set.*

Definition 1.3.4. *The* ***Carathéodory number*** *of a convexity space is the smallest integer c such that whenever $p \in \langle S \rangle$, there exists a subset $F \subseteq S$ with $|F| \leq c$ satisfying*

$$p \in \mathcal{C}(F).$$

Definition 1.3.5. *The* ***Helly number*** *of a convexity space is the smallest integer h such that every family of convex sets with empty intersection contains a subfamily of at most h members whose intersection is empty.*

Definition 1.3.6. *A* ***transit function*** *on a graph G is a mapping*

$$R : V(G) \times V(G) \to 2^{V(G)}$$

satisfying

1. $u \in R(u, v)$,

2. $R(u, v) = R(v, u)$,

3. $R(u, u) = \{u\}$.

Definition 1.3.7. *The* ***induced path transit function*** *of a graph G is*

$$J_G(u, v) = \{x : x \text{ lies on an induced } u\text{–}v \text{ path}\}.$$

The convexity induced by J_G is called the ***induced path convexity*** *or* ***monophonic convexity****.*

Chapter 2

Induced Path Convexity

Graph convexity is a well-established branch of graph theory that extends the classical notion of convexity from Euclidean spaces to discrete structures. In Euclidean geometry, a set is said to be convex if it contains the entire line segment joining every pair of its points. In graph theory, the role of line segments is replaced by suitable classes of paths, leading to various notions of graph convexity. A subset of vertices of a graph is regarded as convex if it contains all the vertices lying on the prescribed paths joining every pair of its vertices. Different choices of path families give rise to different graph convexities, each possessing its own structural and algorithmic properties.

Among the various graph convexities, geodesic convexity and induced path convexity are the most extensively studied. Geodesic convexity is based on shortest paths between vertices, whereas induced path convexity, also known as *monophonic convexity*, is defined using induced (or chordless) paths. An induced path is a path whose induced subgraph contains exactly the edges of the path; equivalently, no two non-consecutive vertices of the path are adjacent. Since induced paths preserve the adjacency structure of graphs, monophonic convexity provides a natural framework for studying several graph classes, including chordal graphs, interval graphs, block graphs, and distance-hereditary graphs.

Graph convexities are usually described by means of interval functions. Let $G = (V, E)$ be a connected graph. An interval function is a mapping

$$R : V(G) \times V(G) \to 2^{V(G)}$$

that assigns to every pair of vertices a set of vertices lying "between" them with respect to a particular family of paths. For example, the geodesic interval function

$$I : V(G) \times V(G) \to 2^{V(G)}$$

is defined by letting $I(u, v)$ denote the set of all vertices lying on shortest u–v paths. Replacing shortest paths by induced paths gives rise to the induced path interval function and hence to induced path convexity.

More generally, let $\mathcal{P}$ be a family of paths in a graph G containing all geodesics. For vertices u and v, the $\mathcal{P}$-interval, denoted by $\mathcal{P}(u, v)$, consists of all vertices lying on a path belonging to $\mathcal{P}$ joining u and v. A subset $S \subseteq V(G)$ is called $\mathcal{P}$-convex if

$$\mathcal{P}(u, v) \subseteq S$$

for every pair of vertices $u, v \in S$. Different choices of $\mathcal{P}$ produce different graph convexities. The most important examples are the all-path convexity, geodesic convexity, triangle-path convexity, and induced path (monophonic) convexity. These convexities may coincide for certain graph classes; for instance, in trees all four convexities are identical.

One of the principal objectives in graph convexity is the study of convexity invariants such as the hull number, Carathéodory number, Helly number, Radon number, and exchange number. These invariants describe different structural properties of convex sets and convex hulls and have been investigated extensively for several graph convexities. In particular, Duchet initiated the systematic study of the Carathéodory and Helly numbers for minimal path convexity, while many subsequent researchers have studied hull sets, monophonic sets, and convexity invariants under induced path convexity.

The fundamental concept in monophonic convexity is the *monophonic interval*. This interval operator induces a convexity on the vertex set of a graph and naturally leads to the notions of monophonic closure, monophonic convex hull, hull sets, and monophonic sets. These concepts form the basis for the study of the convexity invariants discussed in the subsequent chapter.

The objective of this chapter is to introduce the basic concepts and terminology of induced path convexity that will be used throughout this dissertation. We begin with the definition of the monophonic interval and then discuss the associated closure operator, monophonic convex sets, convex hulls, monophonic sets, hull sets, and the monophonic number.

2.1 Basic Definitions

All the definitions in this section are taken from [1]

Definition 2.1.1. *A* ***monophonic path*** *(or* ***induced path****) in a graph G is a path whose induced subgraph is precisely the path itself. Equivalently, a path is induced if no two non-consecutive vertices on the path are adjacent.*

Definition 2.1.2. *Let $u, v \in V(G)$. The* ***monophonic interval*** *between u and v, denoted by*

$$J[u, v],$$

is the set of all vertices lying on at least one induced u–v path in G.

Definition 2.1.3. *For a subset $X \subseteq V(G)$, the* ***monophonic closure*** *of X, denoted by $J[X]$, is defined by*

$$J[X] = \bigcup_{u,v \in X} J[u, v].$$

Thus, $J[X]$ consists of all vertices that lie on an induced path joining some pair of vertices of X.

The operator J is called the *monophonic interval operator*. Repeated application of this operator generates progressively larger vertex sets until no further vertices can be added.

Definition 2.1.4. *A graph G is called **J-monotone** if the interval $J[u,v]$ is J-convex for every pair of vertices $u,v \in V(G)$. Equivalently,*

$$J[x,y] \subseteq J[u,v]$$

whenever $x,y \in J[u,v]$.

In other words, once two vertices belong to the interval $J[u,v]$, every vertex lying on an induced path between them also belongs to $J[u,v]$.

Definition 2.1.5. *A subset $X \subseteq V(G)$ is called **monophonically convex** (or simply m-**convex**) if*

$$J[X] = X.$$

That is, every induced path joining two vertices of X lies entirely within X.

Definition 2.1.6. *The m-**convexity number** of a graph G, denoted by $c_m(G)$, is the maximum cardinality of a proper m-convex subset of $V(G)$.*

Definition 2.1.7. *The smallest m-convex set containing a subset $X \subseteq V(G)$ is called the **monophonic convex hull** of X and is denoted by*

$$J_h[X].$$

Equivalently, $J_h[X]$ is the intersection of all m-convex sets containing X.

Definition 2.1.8. *A subset $X \subseteq V(G)$ is called a **monophonic set** if*

$$J[X] = V(G).$$

If

$$J_h[X] = V(G),$$

*then X is called an m-**hull set**.*

Although every monophonic set is an m-hull set, the converse need not hold, since repeated application of the monophonic closure operator may be required to obtain the entire vertex set.

Definition 2.1.9. *The **monophonic number** of a graph G, denoted by $m(G)$, is the minimum cardinality of a monophonic set of G.*

The concepts introduced in this chapter provide the foundation for the study of the some important convexity invariants associated with induced path convexity. In the next chapter, we investigate these invariants in detail, focusing on the hull number, Carathéodory number and Helly number together with their fundamental properties and known results.

Chapter 3

Convexity Invariants in Induced Path Convexity

In the previous chapter, we introduced the fundamental concepts of induced path (monophonic) convexity, including the monophonic interval, monophonic closure, monophonic convex sets, and the monophonic convex hull. These concepts provide the framework for studying the structural properties of graphs from the viewpoint of convexity. An important direction in graph convexity is the investigation of numerical parameters, known as *convexity invariants*, which measure various aspects of the convex structure induced by a graph.

Convexity invariants play a role analogous to classical graph invariants such as the chromatic number, clique number, and domination number. They quantify different characteristics of the convexity space associated with a graph and provide valuable information about its structural and algorithmic properties. In the context of induced path convexity, several convexity invariants have been extensively studied because of their applications in graph decomposition, optimization, and the characterization of special graph classes.

Among these invariants, the *hull number* is one of the most fundamental. It is defined as the minimum cardinality of a set of vertices whose monophonic convex hull is the entire vertex set of the graph. The hull number measures the smallest number of initial vertices required to generate the whole graph under the monophonic hull operator. Determining the hull number has been the subject of considerable research, and exact values and characterizations have been obtained for several graph classes, including trees, block graphs, interval graphs, and chordal graphs.

Another important invariant is the *Carathéodory number*, which is inspired by the classical Carathéodory theorem in Euclidean convexity. It represents the minimum integer k such that every vertex belonging to the monophonic convex hull of a set can be generated by a subset of at most k vertices. This invariant measures the local generating capacity of convex hulls and provides insight into the complexity of the hull operator.

The *Helly number* is another significant invariant arising from the intersection properties of convex sets. It is the smallest positive integer h such that any finite family of monophonically convex sets satisfying the h-intersection property has a non-empty total intersection. The Helly number reflects the global intersection behaviour of convex sets and establishes a connection between graph convexity and the classical Helly theorem of convex geometry.

These convexity invariants are closely interconnected. While the hull number measures the minimum size of a generating set, the Carathéodory number describes the local generation of vertices within the convex hull, the Helly number characterizes the intersection behaviour of convex sets, and the exchange number concerns the flexibility of minimum hull sets. Together, they offer a comprehensive description of the convexity structure induced by graphs and have motivated extensive research over the past several decades.

The objective of this chapter is to present the definitions, fundamental properties, and important known results concerning these convexity invariants in induced path convexity. We begin with the study of the hull number, followed by the Carathéodory number and the Helly number. Wherever appropriate, illustrative examples and results for important graph classes are included to highlight the behaviour of these invariants and their relationships.

3.1 Hull Number

Definition 3.1.1. ***m-hull number*** *of G, denoted by $hn(G)$, is the cardinality of a minimum m-hull set of G. In other words the hull number is the smallest number of starting vertices from which, by repeatedly including all vertices lying on induced paths between vertices already selected, one eventually obtains every vertex of the graph.*

Example 3.1.1. *For a path graph P_n, the two end vertices form a hull set. Therefore, $hn(P_n) = 2, \qquad (n \geq 2)$.*

Example 3.1.2. *For a complete graph K_n, every induced path has length 1 (only an edge). No additional vertices are generated. Hence every vertex must be chosen initially, hence $hn(K_n) = n$.*

Definition 3.1.2. *A subset $S \subseteq V(G)$ is called J-**convex** if*

$$J[u, v] \subseteq S$$

for every pair of vertices $u, v \in S$. Equivalently, S is J-convex if every induced path between any two vertices of S lies entirely in S.

Theorem 3.1.1. *[3] If a connected graph G is J-monotone, then the hull number of G is the same as the monophonic number of G.*

Proof. Let J be the monophonic interval operator on G. Since G is J-monotone, the operator J is idempotent; that is,

$$J(J(S)) = J(S)$$

for every subset $S \subseteq V(G)$. We first show that every hull set is a monophonic set. Let H be a hull set of G. By definition, there exists a positive integer k such that

$$J^k(H) = V(G),$$

where J^k denotes the k-fold iteration of J. Since J is idempotent,

$$J^2(H) = J(H).$$

By induction, it follows that

$$J^r(H) = J(H)$$

for all $r \geq 1$. Hence,

$$V(G) = J^k(H) = J(H).$$

Therefore, H is a monophonic set. Consequently,

$$m(G) \leq hn(G).$$

Next, let M be a minimum monophonic set of G. Then

$$J(M) = V(G).$$

Since the first application of J already yields the entire vertex set, M is a hull set as well. Therefore,

$$hn(G) \leq m(G).$$

Combining the inequalities

$$m(G) \leq hn(G)$$

and

$$hn(G) \leq m(G),$$

we obtain

$$hn(G) = m(G).$$

Hence, in every connected J-monotone graph, the hull number is equal to the monophonic number. $\square$

Remark 1. *[3] The converse of the above theorem does not hold. For example, the complete bipartite graph $K_{2,3}$ is not J-monotone. However, it is easily verified that $K_{2,3}$ has both hull number and monophonic number equal to 2. Thus, the equality of the hull number and the monophonic number does not imply that a graph is J-monotone. If G is K_n, then the hull number $hn(G)$ is n, since the vertex set of K_n is convexly independent. On the other hand, if G is an atom that is not a clique, then $hn(G) = 2$, since any two non-adjacent vertices of G have the entire vertex set as their convex hull. Thus, we have the following observation.*

Observation 1. *[3] If G is a complete graph k_n, the hull number hn for the induced path convexity of G is n and if G is an atom, which is not a clique, then hn is 2.*

Definition 3.1.3. *A **clique** in G is a set of pairwise adjacent vertices in G. For a vertex x in G, let*

$$N(x) = \{\, y \in V(G) : y \neq x,\ xy \in E(G) \,\}$$

For $W \subseteq V$, define

$$N(W) = \left(\bigcup_{w \in W} N(w) \right) \setminus W.$$

Definition 3.1.4. *Let G be a graph. A set $C \subseteq V(G)$ is a **clique separator** of G if C is a complete set and $G - C$ is disconnected.*

Definition 3.1.5. *A set S is a **minimal separator** in G if and only if there exist two distinct components Let $C_1, C_2 \in C(S)$. such that*

$$N(C_1) = N(C_2) = S.$$

*The components C_1 and C_2 are then called the full components of $C(S)$. A set S is a minimal a–b separator if and only if a and b belong to two different full components of $C(S)$. S is called a **minimal clique separator** if S is a clique and a minimal separator.*

Definition 3.1.6. *A graph is called an* ***atom*** *if it has no clique seperator.*

Definition 3.1.7. *The* ***atom–clique separator tree*** *(also called a decomposition tree) is a tree that represents how a graph is recursively split into atoms using clique separators.*

Remark 2. *[3] Note that a subgraph G(A) has no clique separator if and only if G(A) has no minimal clique separator: as any separator contains some minimal separator. Thus if A has a clique separator, then it has a minimal clique separator*

Using this remark, we define the following intersection graph on G. The atom–clique separator tree $T(G)$ of G is defined as follows. The vertices of $T(G)$ are the atoms and minimal clique separators of G. The edges of $T(G)$ are defined as follows: if A is an atom and S is a minimal clique separator such that S separates the vertices of A from the vertices of $G(V \setminus (S \cup A))$ and $N(A) = S$. It can be verified that $T(G)$ is a tree. For if it contains a cycle, say C, then there exists a minimal clique separator in C whose removal does not disconnect G, a contradiction.

Let G be a connected graph which is neither complete nor an atom. Let $T(G)$ denote the atom–clique separator tree of G such that every leaf of $T(G)$ is not a clique. Let $L(T(G))$ denote the set of leaves of $T(G)$. We have the following proposition:

Proposition 3.1.1. *[3] Let G be a connected graph which is neither an atom nor a complete graph K_n, such that every leaf of $T(G)$ is not a clique. Then the hull number hn for the induced path convexity of G is given by*

$$hn = |L(T(G))|.$$

Proof. By the definition of $T(G)$, every leaf of $T(G)$ represents an atom of G, and the atom corresponding to a leaf of $T(G)$ is not a clique. Corresponding to every element $a' \in L(T(G))$, there exists an atom A of G; choose a representative vertex, say $a \in A$. Let $S \subseteq V(G)$ consist of representative vertices chosen from the atoms corresponding to every element of $L(T(G))$. Clearly,

$$|S| = |L(T(G))|.$$

We have to show that every vertex x of V lies in the convex hull of two vertices, say u and v of S, which corresponds to two distinct atoms A and B, respectively, in $L(T(G))$. Also we have to show that S is a set of minimum cardinality satisfying the above condition.

We first show the minimality of S. For this, we prove that every element $a \in S$ is necessary in any hull set of G. Since a belongs to an atom of G, which corresponds to a leaf of $T(G)$, there exists a minimal clique separator that separating a and $G \setminus a$. Therefore, a does not belong to any induced path between a pair of vertices in $G \setminus a$. Consequently,

$$\Big\langle J\big(V(G \setminus a)\Big\rangle$$

does not contain any vertex of A. Therefore a must belong to the hull set of G. This shows that the set S is convexly independent and must belong to the hull set of G. Now, we prove that S is a hull set. Suppose not. That is,

$$\langle S \rangle \neq V(G).$$

Then there exists a vertex $x \in V(G)$ such that

$$x \notin \langle S \rangle.$$

Since $x \notin \langle S \rangle$, there exists a clique separator K seperating x and $\langle S \rangle$. Without loss of generality, K can be taken as a minimal clique seperator. If

$$x \notin \langle S \cup K \rangle,$$

We get another minimal clique seperator K_1 seperating x and $\langle S \cup K \rangle$. Continuing like this,we can get a sequence of minimal clique seperators $K_1, K_2, ..., K_m$ such that

$x \in \langle S \cup K \cup K_1 \cup \cdots \cup K_m \rangle$. If $\langle S \cup K \cup K_1 \cup \cdots \cup K_m \rangle \neq V$ then we can replace $S \cup K \cup K_1 \cup \cdots \cup K_m$ by S and then there exist an x_1 such that $x_1 \notin \langle S \rangle$. Arguing as before, since G is finite, we obtain a vertex x_n and a sequence of minimal clique separators $L_1, L_2, \ldots, L_r$ such that

$$x_n \notin \langle S \cup L_1 \cup L_2 \cup \cdots \cup L_{r-1} \rangle$$

and

$$x_n \in \langle S \cup L_1 \cup L_2 \cup \cdots \cup L_r \rangle = V(G).$$

This shows that x_n belongs to an atom corresponding to a leaf of $T(G)$, a contradiction. Therefore, the set S corresponding to $L(T(G))$ is a hull set. Thus S is both convex independent and a hull set. Thus S is a convex independent as well as a hull set and so is a minimal hull set and hence the hull number is exactly

$$|S| = |L(T(G))|.$$

□

Suppose one of the atoms A corresponding to a leaf of $T(G)$ is itself a clique. Let S be the minimal clique separator separating A from the rest of G. If the clique $A \setminus S$ is of size $m \geq 2$, then we add m new vertices to the graph $T(G)$ and join each of them by edges from the vertex corresponding to A in $T(G)$. We repeat this procedure for every clique atom corresponding to a leaf in $T(G)$, and the resulting tree is denoted by $T'(G)$. Denote the set of leaves of $T'(G)$ by $L(T'(G))$. For each $x \in L(T'(G))$, we take a corresponding vertex in $V(G)$ as before, and the set of vertices thus obtained forms a convex-independent set of $V(G)$ with respect to the induced path convexity.

Proposition 3.1.2. *[3] Let G be a connected graph which is not an atom and different from K_n. For the induced path convexity of G, the hull number hn satisfies*

$$hn(G) = |L(T'(G))|.$$

Proof. Let

$$L(T'(G)) = \{\ell_1, \ell_2, \ldots, \ell_k\},$$

where $k = |L(T'(G))|$.

For each leaf ℓ_i of $T'(G)$, choose a corresponding vertex $x_i \in V(G)$ as follows:

- If ℓ_i corresponds to a non-clique leaf atom A, choose a simplicial vertex of A.
- If ℓ_i is one of the new leaves added for a clique atom A, choose the corresponding vertex of $A \setminus S$.

Let

$$X = \{x_1, x_2, \ldots, x_k\}.$$

By construction, X is a convex-independent set with respect to the induced-path convexity. We first show that

$$hn(G) \geq |L(T'(G))|.$$

Let A be a leaf atom separated from the rest of the graph by a minimal clique separator S. Every vertex of $A \setminus S$ belongs to a component of $G - S$ that is disjoint from the remaining leaf atoms. Consequently, no induced path joining vertices outside this component can generate a vertex of $A \setminus S$. Hence every hull set must contain at least one representative corresponding to each leaf of $T'(G)$. Therefore,

$$hn(G) \geq k = |L(T'(G))|.$$

Next, we show that X is a hull set. Consider a leaf atom A and let S be the minimal clique separator adjacent to it. Since X contains a representative vertex from every leaf of $T'(G)$, there exist vertices of X lying in different components determined by every minimal clique separator of the atom decomposition. A standard property of monophonic convexity implies that every vertex of such a separator S lies on an induced path between suitable vertices of X. Hence,

$$S \subseteq J_m(X),$$

where $J_m(X)$ denotes the monophonic interval of X. Once the vertices of S belong to the hull, the atom adjacent to A becomes absorbed into the convex hull. Repeating this argument along the edges of the tree $T'(G)$, every separator and every atom of the decomposition is successively included in the iterated monophonic hull of X. Since $T'(G)$ is a tree, every atom of G is eventually reached from the leaves. Therefore,

$$J_h[X] = V(G),$$

where $J_h[X]$ denotes the monophonic hull of X. Thus X is a hull set of G, and hence

$$hn(G) \leq |X| = |L(T'(G))|.$$

Combining the two inequalities,

$$|L(T'(G))| \leq hn(G) \leq |L(T'(G))|,$$

we obtain

$$hn(G) = |L(T'(G))|.$$

Therefore, the hull number of G with respect to the induced-path convexity is equal to the number of leaves of the modified atom tree $T'(G)$. □

Theorem 3.1.2. *[1] Let G be a graph. A subset $X \subseteq V(G)$ is m-convex if and only if for every pair of non-adjacent vertices $u, v \in X$ and every connected component C of $G - X$ it hold either $V(C) \cap N(u) = \varnothing \quad or \quad V(C) \cap N(v) = \varnothing$.*

Proof. Assume that X is m-convex. The existence of a pair of non-adjacent vertices $u, v \in X$ and a connected component C of $G - X$ containing a vertex adjacent to both u and v, such that $V(C) \cap N(u) = \varnothing \quad \text{or} \quad V(C) \cap N(v) = \varnothing$ implies the existence of a sequence of vertices $w_0 = u,\ w_1 = u',\ w_2, \ldots, w_{k-1},\ w_{k+1} = v$ such that:

1. $w_i \notin X$, $1 \leq i \leq k$;

2. either $u' = v'$ or $(w_i w_{i+1}) \in E(G - X)$ for $1 \leq i \leq k$

Hence, there exists an induced path linking u and v containing at least one vertex outside X, a contradiction.

Now assume that X is not m-convex. Let $w_0 = u,\ w_1 = u',\ w_2, \ldots, w_{k-1},\ w_{k+1} = v$ be an induced path linking a pair of vertices $u, v \in X$ such that $k \geq 1$ and $w_i \notin X$ for some i. Let j be an index such that $w_{j-1} \in X$ and

$$w_j, w_{j+1}, \ldots, w_i \in V(G) \setminus X$$

(such an index exists since $u \in X$). Similarly, let ℓ be an index such that

$$w_i, w_{i+1}, \ldots, w_l \in V(G) \setminus X$$

and $w_{\ell+1} \in X$. This implies that w_{j-1} and $w_{\ell+1}$ are a pair of non-adjacent vertices in X, and there exists a connected component C of $G - X$ such that

$$V(C) \cap N(w_{j-1}) \neq \varnothing \quad \text{and} \quad V(C) \cap N(w_{\ell+1}) \neq \varnothing.$$

This completes the argument. $\square$

Theorem 3.1.3. *[1] If G is an atom that is not a complete graph, then every pair of non-adjacent vertices is an m-hull set of G.*

Proof. Let $x, x' \in V(G)$ be two non-adjacent vertices and $S = J_h[\{x, x'\}]$. Suppose that $S' = V(G) \setminus S \neq \emptyset$. Consider a maximal subset $C \subseteq S$ satisfying the following properties:

1. C is a complete set;

2. there exists a connected subgraph G' of $G[S']$ such that every $y \in C$ has a neighbor $z \in V(G')$

It is clear that $C \neq \emptyset$. Since G does not contain clique separators, the graph $H = G - C$ is connected. Then H contains an induced path

$$z_1, z_2, \ldots, z_k \quad (k \geq 2),$$

such that $z_1 \in V(G')$, $\{z_2, \ldots, z_{k-1}\} \subseteq S'$ and $z_k \in S$. The vertex z_k exists because S is not a complete set. Since C is maximal, there exists $y \in C$ which is not adjacent to z_k; otherwise $C \cup \{z_k\}$ and $G[V(G') \cup \{z_2, \ldots, z_{k-1}\}]$ would satisfy conditions (1) and (2) , respectively. Let $z \in V(G')$ be a neighbor of y. Since there exists a path in G' linking z and z_1, we conclude that there exists an induced path linking y and z_k whose internal vertices lie outside S, a contradiction. □

Lemma 3.1.1. *[1] Let G be a graph, C a clique separator of G, and let V' be the union of the vertex sets of some connected components of $G - C$. Then $V' \cup C$ is an m-convex set of G.*

Proof. Suppose the contrary. Then $V' \cup C$ contains two vertices u, v in different components of $G - C$ such that $P = uv_1 \ldots v_k v$, $k \geq 1$, is an induced path containing a vertex v_j $(1 \leq j \leq k)$ lying outside $V' \cup C$. Let p (resp. r) be the minimum (resp. maximum) index such that $v_p \in C$ (resp. $v_r \in C$). Then $1 \leq p < j < r \leq k$, and since C is a clique separator of G, we have $(v_p, v_r) \in E(G)$. This is a contradiction. □

3.2 Carathéodory Number

The **Carathéodory number** is an important parameter in convexity theory that measures how many points are sufficient to represent any point in a convex hull. In a general convexity space, the Carathéodory number is the smallest positive integer k such that for every set X and every point $x \in \langle X \rangle$, there exists a subset $S \subseteq X$ with $|S| \leq k$ such that $x \in \langle S \rangle$. In other words, every element of the convex hull of a set can be generated by a convex combination of at most k elements of that set. In classical Euclidean space $\mathbb{R}^d$, Carathéodory's theorem shows that this number is $d + 1$. In graph convexity settings such as monophonic convexity, the concept is adapted by replacing convex combinations with convexity defined through induced paths, and the Carathéodory number similarly captures the maximum size of a subset needed to generate any vertex in the monophonic convex hull of a set.

Definition 3.2.1. *Let $S \subseteq X$ be a nonempty finite set. The set S is **Carathéodory dependent** (or, C-dependent) provided*

$$\langle S \rangle \subseteq \bigcup_{a \in S} \langle S \setminus \{a\} \rangle$$

*and it is **Carathéodory independent** (or, C-independent) otherwise. That is, a subset S of X is said to be C-independent, if there is*

$$p \in \langle S \rangle \setminus \bigcup_{a \in S} \langle S \setminus \{a\} \rangle$$

Definition 3.2.2. *A set $A \subseteq V$ is **redundant** when it is not empty and has the property*

$$\langle A \rangle = \bigcup_{a \in A} \langle A \setminus \{a\} \rangle$$

Lemma 3.2.1. *[2]. In any convexity space, the Carathéodory number equals the maximum cardinality of an irredundant set.*

Definition 3.2.3. *The function I is called an **interval-function** of the convexity space (V, C) when*

$$\langle A \rangle = \bigcup_{k \in \mathbb{N}} I^k(A)$$

where I^k is defined by $I^0(A) = A$ and $I^{k+1}(A) = I\left(I^k(A) \times I^k(A)\right)$.

Definition 3.2.4. *The least integer k such that a given element x of $\langle A \rangle$ belongs to $I^k(A)$ is called the **index of x relative to A and I**.*

Definition 3.2.5. *Convexity spaces admitting interval functions are named **interval convexity spaces**.*

Lemma 3.2.2. *[2]. In an interval convexity space, the Carathéodory number is the smallest integer such that every (c+1)-element set is redundant.*

Proof. Let (V, C) be an interval convexity space. We suppose that for some integer c, every $(c+1)$-element subset of V is redundant but V contains irredundant subsets of each cardinality $\leq c$. By Lemma 3.2.1, the Carathéodory number is atleast c. In order to prove the equality, let us consider a V-subset A and an element $x \in \langle A \rangle$. We have to find a subset B of A such that $x \in \langle B \rangle$ and $|B| \leq c$. Consider all subsets B of $\langle A \rangle$ such that $x \in \langle B \rangle$ and $|B| \leq c$. Since $x \in \langle x \rangle$ and $x \in \langle A \rangle$, this collection of subsets is not empty. Pick a B so as to minimize the sum

$$\sigma(B) = \sum_{b \in B} 3^{i(b)}$$

where i(b) denotes the index of b relative to the set A and to the interval function I. If $B \subseteq A$, we are done. If not, let b be an element of $B \setminus A$, with index $k \geq 1$. By formula

$$\langle A \rangle = \bigcup_{k \in \mathbb{N}} I^k(A)$$

there are elements v, w of $I^{k-1}(A)$ such that b $\in I(v, w)$. Let $B' = B \cup \{v, w\} \setminus \{b\}$. The $x \in \langle B \rangle \subseteq \langle B' \rangle$. Moreover, $|B'| \leq |B| + 1 \leq c + 1$. Since $i(v) \leq k - 1$ and $i(w) \leq k - 1$ and $3^n > 3^{n-1} + 3^{n-1}$, it follows that $\sigma(B') < \sigma(B)$. The choice of B implies $|B'| = c+1$, so by hypothesis B' is redundant and contains a set B'' with $|B''| = c$ and $x \in \langle B'' \rangle$. But $\sigma(B'') \leq \sigma(B') < \sigma(B)$, contradicting our choice of B. $\square$

Remark 3. *Because of Lemma 3.2.1 an equivalent statement of Lemma 3.2.2 is the following: in any interval convexity space, the cardinalities of irredundant sets form an interval $[0, c]$ of $\mathbb{N}$; this property fails for general convexity spaces. For instance the convexity on a set V constituted by all subsets of cardinality $\leq n$ has Carathéodory number $n+1$ but every $k-$ element set is redundant for $2 \leq k \leq n$.*

Definition 3.2.6. *Let $G = (V, E)$ be a connected graph which may be infinite and the set of all vertices on all chordless (x, y)-path in G is denoted by $M(x, y)$. Operator M is an interval-function for M-convexity space. Then M-convex hull in G is simply denoted by $\langle \ \rangle$; we have*

$$\langle x, y \rangle = \bigcup_{k \in \mathbb{N}} M^k(x, y)$$

Special notations are used for *walks* in G: a walk from x to y, an (x, y)-walk for short, is a sequence μ of vertices denoted by simple juxtaposition—for instance, $\mu = v_1 v_2 .... v_p$ such that $v_1 = x$, $v_p = y$ and v_{i+1} is adjacent to v_i for $1 \leq i \leq p$. If no confusion can arise (for instance, when vertices of μ occur only once in μ, i.e., μ is a path), the expression $v_r \overset{\mu}{—} v_s$, with $1 \leq r \leq s \leq p$, denotes the walk from v_r to v_s extracted from μ. The concatenation μv of two walks $x \overset{\mu}{—} y$ and $y \overset{v}{—} z$ is denoted by $x \overset{\mu}{—} y \overset{v}{—} z$.

Theorem 3.2.1. *[2]. Let G=(V,E) be a connected graph which is not a clique. Then the minimal path convexity has Carathéodory number 2.*

Proof. By Lemma 3.2.2, it suffices to prove that every 3-element set is redundant,i.e, For

$$a, b, c \in V, \quad \langle a, b, c \rangle = \langle a, b \rangle \cup \langle b, c \rangle \cup \langle c, a \rangle$$

The proof of this equation goes by induction on the index of elements of $\langle a, b, c\rangle$ and uses two technical lemmas:

Lemma 3.2.3. *Let* a,b,c *be three vertices of* G *and* $x \in M(a, b)$. *Then we have*

$$M(x, c) \subseteq \langle a, b\rangle \cup M(a, c) \cup M(b, c)$$

Proof. Suppose some $y \in M(x, c)$ does not belong to $\langle a, b\rangle \cup M(a, c) \cup M(b, c)$. Then the vertices a, b, c, x, y are all different. Choose a minimal (a, b)-path μ that contains x and a minimal (x, c)-path v that contains y. Since $y \notin M(a, c)$, the walk $a \overset{\mu}{—} x \overset{v}{—} c$ contains a minimal (a, c)-path α that avoids y. Similarly, the walk $b \overset{\mu}{—} x \overset{v}{—} c$ contains a minimal (b, c)-path β that avoids y. Because v is a minimal (x, c)-path, the path α and β also avoid x. Hence α equals $a \overset{\mu}{—} a' c_\alpha \overset{v}{—} c$ and β equals $b \overset{\mu}{—} b' c_\beta \overset{v}{—} c$ for some vertices $a', c_\alpha, b', c_\beta$. The paths $a \overset{\mu}{—} a'$ and $b \overset{\mu}{—} b'$ are disjoint and do not contain x.

Without loss of generality, we may suppose c_α occurs in the path $c_\beta \overset{v}{—} c$. Therefore, $c_\alpha \overset{\beta}{—} c_\beta$ and $c_\alpha \overset{v}{—} c_\beta$ coincide: the walk $a \overset{\alpha}{—} a' c_\alpha \overset{\beta}{—} c_\beta b' \overset{\beta}{—} b$ equals $a \overset{\mu}{—} a' c_\alpha \overset{v}{—} c_\beta b' \overset{\mu}{—} b$. This walk is an (a, b)-path which contains a minimal (a, b)-path π. By minimality of μ, no edge joins $a \overset{\mu}{—} a'$ to $b' \overset{\mu}{—} b$ and so π has some vertex c' in $c_\alpha \overset{v}{—} c_\beta$. By definition of α and β, the vertex y in on $c_\alpha \overset{v}{—} x$ and on $c_\beta \overset{v}{—} x$. Hence y is on $c' \overset{v}{—} x$. Since both x and c' are in $\langle a, b\rangle$, so is y and hence lemma is proved. $\square$

Lemma 3.2.4. *Let* a, b, c *be three vertices of* G *and* $x \in \langle a, b\rangle$. *Then*

$$M(x, c) \subseteq \langle a, b\rangle \cup M(a, c) \cup M(b, c)$$

holds.

Proof. By induction on the index k of x relative to $\{a, b\}$. Set $A = \langle a, b\rangle \cup M(a, c) \cup M(b, c)$. If $k \leq 1$, Lemma 3.2.3 above applies. If $k > 1$, we have $x \in M(x', x'')$ for some vertices $x', x'' \in M^{k-1}(a, b)$. Applying Lemma 3.2.3 to the triple x', x'', c we have

$$M(x, c) \subseteq (x', x'') \cup M(x', c) \cup M(x'', c)$$

By induction hypothesis A contains both sets $M(x', c)$ and $M(x'', c)$. Since $\langle x', x''\rangle \subseteq \langle a, b\rangle$, we have $M(x, c) \subseteq A$. $\square$

Proof. The proof of $a, b, c \in V, \quad \langle a, b, c\rangle = \langle a, b\rangle \cup \langle b, c\rangle \cup \langle c, a\rangle$. Let $y \in \langle a, b, c\rangle$. We prove $y \in \langle a, b\rangle \cup \langle b, c\rangle \cup \langle c, a\rangle$ by induction on the index k of y relative to $\{a, b, c\}$. The assertion is trivial if $k \leq 1$. If $k > 1$, we have $y \in M(y', y'')$ for some vertices $y', y'' \in M^{k-1}(a, b, c)$. Applying the induction hypothesis to y' and y'', we may suppose without loss of generality that $y' \in \langle a, b\rangle$ and $y'' \in \langle b, c\rangle$. By Lemma 3.2.4 (with y'' instead of c and y' instead of x) we obtain

$$M(y', y'') \subseteq \langle a, b\rangle \cup M(a, y'') \cup M(b, y'')$$

We have $M(b, y'') \subseteq \langle b, c\rangle$. Moreover by Lemma 3.2.4 again, we have $M(y'', a) \subseteq \langle b, c\rangle \cup M(b, a) \cup M(c, a)$. Hence the set $\langle a, b\rangle \cup \langle b, c\rangle \cup \langle c, a\rangle$ contains $M(y', y'')$ and a fortiori,y.

$\square$

3.3 Helly Number

Definition 3.3.1. *A family $\mathcal{F}$ of sets has the k-intersection property (k-I.P.), where k is a positive integer, if every k or fewer sets in $\mathcal{F}$ have a nonvoid (nonempty) intersection. The* ***Helly number*** *of the family $\mathcal{F}$ is the smallest positive integer h such that any finite subfamily of $\mathcal{F}$ with the h-intersection property has a nonvoid intersection.*

Definition 3.3.2. *Let $G = (V, E)$ be a graph and let $S \subseteq V$. Two vertices $s, t \in S$ are said to be* ***joined externally*** *to S if there exists an s–t path whose internal vertices do not belong to S; that is, there exists a path*

$$P = s, x_1, x_2, \ldots, x_n, t$$

such that

$$x_i \notin S, \qquad \textit{for } i = 1, 2, \ldots, n.$$

Lemma 3.3.1. *[4] A set S of vertices in G is* m-convex *iff any pair of vertices in S which are joined externally to S are adjacent.*

Proof. Suppose S is m-convex and let

$$P = s - x_1 - x_2 - \cdots - x_n - t$$

be a path with $s, t \in S$ and $x_i \notin S$ for all i. Among all such paths, choose a shortest one. Such a path need not be a shortest path in G, but it has no chords (otherwise we could obtain a shorter path), except possibly $s - t$. Hence if s and t are not adjacent, then P is a chordless path. Since S is m-convex, every chordless path joining two vertices of S must lie entirely in S. And since $s - t$ is not an edge, there must be atleast one x_i, contradicting $x_i \notin S$. Thus s and t must be adjacent.

Conversely, suppose that every pair of vertices joined externally to S are adjacent. Let P be a chordless path joining two vertices of S. We must show that $P \subseteq S$. Assume, to the contrary, that P contains a vertex $x \notin S$. Let s be the last vertex of S on P before x, and let t be the first vertex of S on P after x. Then the subpath of P from s to t joins s and t externally to S. By hypothesis, s and t must be adjacent. Hence the edge $s - t$ is a chord of P, contradicting the assumption that P is chordless. Therefore, every vertex of P belongs to S, and so $P \subseteq S$. Thus, S is m-convex. □

Lemma 3.3.2. *[4] Suppose $\mathcal{L}$ is a family of sets closed under intersection. If, for some positive integer k, any finite family $\mathcal{F}$ of sets in $\mathcal{L}$ with the k-intersection property also satisfies the $(k+1)$-intersection property, then the Helly number of $\mathcal{L}$ is at most k.*

Proof. We show that any finite family $\mathcal{F}$ of sets in $\mathcal{L}$ that has the k-intersection property also has the n-intersection property for all $n \geq k$. This is true by hypothesis for $n = k+1$. Proceeding by induction on n, let $A_0, A_1, ..., A_n$ be sets in $\mathcal{F}$ and define

$$\mathcal{A}^* = \{A_i \cap A_0 : i \neq 0\}$$

Since F has the $k+1$-intersection property, $\mathcal{A}^*$ has the k intersection property $\mathcal{F}$ is closed under intersection, $\mathcal{A}^*$ and hence the n-intersection property by induction. Thus the intersection of n sets in $\mathcal{A}^*$ is nonempty, that is,

$$A_1 \cap A_2 \cap \cdots \cap A_n \neq \emptyset.$$

Hence, $\mathcal{F}$ has the $n+1$-intersection property. □

Theorem 3.3.1. *[4] For any connected graph G, the helly number of m-convex sets equals the clique number of G.*

Proof. Let k be the clique number of G, and let K be a clique of k vertices. Clearly K and all its subsets are trivially m-convex. Since the k subsets of K with $k-1$ vertices form a family with the $k-1$ intersection property but empty intersection, the Helly number is at least k. From the definition of m-convexity, the family of m-convex sets is closed under intersection. Hence, to show that the Helly number is at most k, it suffices by Lemma 3.3.2 to show that the k-intersection property implies the $(k+1)$-intersection property for m-convex sets.

Let $A_0, A_1, A_2, \ldots, A_k$ be m-convex sets such that any k of them have nonempty intersection. For each i, let

$$D_i = \bigcap_{j \neq i} A_j,$$

so each D_i is a nonempty m-convex set. Suppose

$$\bigcap_{i \geq 0}^{k} A_i = \varnothing, \quad \text{equivalently } A_i \cap D_i = \varnothing \text{ for each } i.$$

Select points $x_i \in D_i$ in such a way as to maximize the cardinality of the largest clique in

$$S = \{x_0, x_1, \ldots, x_k\}$$

With no loss in generality, we may assume the indices so chosen that $x_0, x_1, \ldots, x_k$ is a clique of largest cardinality in S. Since k is the clique number, we should have $n < k$. We aim to contradict this

Case 1: $n = 0$. Certainly $k \geq 0$ Since G is connected, there exist vertices x_1, x_2. Let P and Q be shortest paths from x_2 to x_0 and from x_2 to x_1, respectively. Since $x_i \in D_i$ and $A_i \cap D_i = \varnothing$ for each i, both x_0 and x_1 lie in A_2 but x_2 does not. Let y be the last point of P in A_2. Since P is shortest, it is certainely chordless and hence lies in any m-convex set containing x_2 and x_0 . In particular, P and hence y lies in A_i for $i \neq 0, 2$. But $y \in A_2$ by choice. Hence $y \in D_0$. Likewise $z \in D_1$. Hence if we replace x_0 by y and x_1 by z, we get a new set

$$S' = (S \setminus \{x_0, x_1\}) \cup \{y, z\}$$

satisfying the choice criterion of one point from each D_i. Since y and z are joined externally to A_2 along P and Q via x_2, and since A_2 is m-convex, Lemma 3.3.1 implies that y and z are adjacent. Thus S' contains a larger clique than S, a contradiction.

Case 2: $n > 0$. Assume $x_1, x_2, \ldots, x_{n+1} \in A_0$ while $x_0 \notin A_0$. Let P be a shortest path from x_0 to x_{n+1} and let y be the first vertex of A_0 on P. For each i from 1 to n, x_i and y are joined externally to A_0 via the edge $x_i - x_0$ and the subpath of P from x_0 to y. By m-convexity and Lemma 3.3.1, y is adjacent to all x_i for $1 \leq i \leq n$. As in case 1, $y \in D_{n+1}$ since it is in A_0 by choice and P lies in A_i for $i \neq 0,\ n+1$ by m-convexity. Now x_1 is adjacent to x_0 and y.(It is here that we use $n \geq 1$ to be sure that y is not replacing x_1). Since x_0 and y lie in A_1 but x_1 does not, the path $x_0 - x_1 - y$ joins x_0 and y externally to A_1. Thus x_0 and y must be adjacnet by Lemma 3.3.1. But then $x_0, x_1, \ldots.., x_n$, y is a clique. Thus, since

$$S' = (S \setminus \{x_{n+1}\}) \cup \{y\}$$

is a legitimate choice of representatives for the D_i, we have increased the largest clique size in such. This contradiction completes the proof. □

Corollary 3.3.1. *[4] Let F be a finite family of $m-$ convex sets in a graph G. If F has the K-intersection property and some set A in F contains no clique on k vertices, then the sets in F have some vertices in common.*

Proof. The family

$$A^* = \{A \cap B : B \in \mathcal{F}\}$$

is a family of m-convex subsets of A with the k-intersection property. By hypothesis, the clique number of A is atmost $k-1$. Thus the corollary follows by applying Theorem 3.3.1 to the subgraph induced by A. □

Definition 3.3.3. *The set S is* ***Helly dependent*** *(or,H-dependent) provided*

$$\bigcap_{a \in S} \langle S \setminus \{a\} \rangle \neq \varnothing$$

and it is ***Helly independent*** *(or, H-independent) otherwise.*

Lemma 3.3.3. *[2] In any convexity space, the helly number is the least integer h such that every (h+1)-element set $A \subseteq V$ has the property*

$$\bigcap_{a \in A} \langle A \setminus \{a\} \rangle \neq \varnothing$$

Lemma 3.3.4. *[2] If a convexity space has helly number h and randon number r, then $h+1 \leq r$.*

Conclusion

This project presented a detailed study of induced path convexity and its associated convexity parameters, namely the hull number, Carathéodory number, and Helly number. These parameters play a fundamental role in understanding the convex structure of graphs and provide useful tools for analyzing various graph classes under induced path convexity.

The study of the hull number focused on its behavior in trees, emphasizing the role of leaves in determining minimum hull sets. The influence of clique separators on the computation and characterization of hull sets was also examined, illustrating how graph decomposition can simplify the analysis of convexity properties. These results demonstrate the close relationship between graph structure and hull generation.

The investigation of the Carathéodory number centered on the concepts of redundant sets and joined externally sets. These notions provide a deeper understanding of minimal generating sets in induced path convexity and offer criteria for identifying essential vertices in convex representations. The results highlight how redundancy and external connectivity influence the formation of convex hulls and the determination of Carathéodory sets.

The project also discussed the Helly number through the framework of the (k+1)-intersection property, illustrating the conditions under which families of convex sets possess a common intersection. This establishes an important connection between local intersection properties and global convexity, further enriching the theory of induced path convexity.

Overall, this project demonstrates that induced path convexity provides a powerful framework for studying graph convexity through different yet interconnected parameters. The concepts of hull number, Carathéodory number, and Helly number collectively reveal important structural characteristics of graphs and contribute to a deeper understanding of convexity in graph theory. The results presented here also suggest several directions for future research, including the study of these parameters in broader graph classes, the development of efficient computational algorithms, and the exploration of applications in network analysis, combinatorial optimization, and related areas of discrete mathematics.

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