AN INTRODUCTION TO RIEMANNIAN MANIFOLDS

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MASTER OF SCIENCE IN MATHEMATICS

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CERTIFICATE

This is to certify that the dissertation entitled **"An Introduction to Riemannian Manifolds"** is a record of studies carried out by SAMUDRA PRASAD, M.Sc. Mathematics student of the year **2024–2026** at **T.K.M. College of Arts and Science, Kollam**, under my guidance and submitted to the University of Kerala in partial fulfillment of the Degree of Master of Science in Mathematics.

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DECLARATION

I hereby declare that the dissertation entitled **"An Introduction to Riemannian Manifolds"** is a record of studies carried out by me under the guidance of Ms. AGASTHYA KRISHNAN A , Assistant Professor, Department of Mathematics, T.K.M. College of Arts and Science, Kollam and has not been submitted by me previously for the award of any degree.

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Contents

INTRODUCTION

This project introduces the basic ideas in topology and differential geometry, with a focus on topological manifolds and smooth structures. It explains important definitions, theorems, and examples to help understand spaces that look like Euclidean space near each point.

Chapter 1, *Preliminaries*, gives the necessary background. It covers topics like topological spaces, countability, and inner product spaces. These ideas are important for understanding the rest of the project.

Chapter 2 explains *topological manifolds.* These are spaces that are locally similar to $\mathbb{R}^n$. It talks about properties like being Hausdorff, second-countable, and locally Euclidean. It also discusses charts and coordinate systems.

Chapter 3 introduces *smooth manifolds.* These are topological manifolds with extra structure that allows us to do calculus. It defines smooth maps and gives examples such as spheres and projective spaces.

Chapter 4 talks about *tangent vectors and tangent bundles.* These help us study directions and changes on a manifold. The chapter explains how we define tangent vectors and how they are related to differentials.

Chapter 5 looks at *Riemannian manifolds.* These are smooth manifolds where each tangent space has an inner product. This helps us measure angles, lengths, and curvature on the manifold.

Overall, this project gives a basic and clear understanding of manifolds and their structure, which are important in many areas of mathematics and physics.

Chapter 1

PRELIMINARIES

1.1 Topological Spaces

Let X be a set. A *topology* on X is a collection $\mathcal{T}$ of subsets of X satisfying the following properties:

1. $X \in \mathcal{T}$ and $\emptyset \in \mathcal{T}$.
2. $\mathcal{T}$ is closed under finite intersections: If $U_1, \ldots, U_n \in \mathcal{T}$, then the intersection
$$U_1 \cap U_2 \cap \cdots \cap U_n \in \mathcal{T}.$$
3. $\mathcal{T}$ is closed under arbitrary unions: If $\{U_\alpha\}_{\alpha \in A}$ is any (finite or infinite) family of elements of $\mathcal{T}$, then the union
$$\bigcup_{\alpha \in A} U_\alpha \in \mathcal{T}.$$

A pair $(X, \mathcal{T})$ consisting of a set X together with a topology $\mathcal{T}$ on X is called a *topological space.*

Definition 1.1. Let X be a topological space and let $p \in X$. A *neighborhood* of p is an open subset of X that contains p.

Example 1.2. (Simple Topologies)

(a) Let X be any set whatsoever, and let $\mathcal{T} = \mathcal{P}(X)$ (the power set of X, which is the set of all subsets of X), so every subset of X is open.This is called the *discrete topology* on X, and $(X, \mathcal{T})$ is called a *discrete space.*

(b) Let Y be any set, and let $\mathcal{T} = \{Y, \varnothing\}$. This is called the *trivial topology* on Y.

Example 1.3. (The Metric Topology) Let (M, d) be any metric space, and let $\mathcal{T}$ be the collection of all subsets of M that are open in the metric space sense. This is a topology, called the *metric topology* on M, or the *topology generated by d.*

Definition 1.4.

The unit interval is the subset $I \subset \mathbb{R}$ defined by

$$I = [0, 1] = \{x \in \mathbb{R} \mid 0 \leq x \leq 1\}.$$

For any nonnegative integer n, the **(open) unit ball** of dimension n is the subset $B^n \subset \mathbb{R}^n$ consisting of all vectors of length strictly less than 1:

$$B^n = \{x \in \mathbb{R}^n \mid \|x\| < 1\}.$$

In the case $n = 2$, we sometimes call B^2 the *(open) unit disk.*

The **closed unit ball** of dimension n is the subset $\overline{B}^n \subset \mathbb{R}^n$ consisting of all vectors of length at most 1:

$$\overline{B}^n = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}.$$

We sometimes call $\overline{B}^2$ the *closed unit disk.*

Definition 1.5. If X is a topological space, a subset $F \subset X$ is said to be a closed subset of X if its complement $X \setminus F$ is an open subset. If X and its topology are understood, closed subsets of X are often just called **closed sets**.

Definition 1.6. The **closure** of a subset $A \subset X$, denoted by $\overline{A}$, is defined as:

$$\overline{A} = \bigcap\{B \subset X \mid B \supset A \text{ and } B \text{ is closed in } X\}.$$

Definition 1.7. The **interior** of A, denoted by $\operatorname{Int} A$, is:

$$\operatorname{Int} A = \bigcup \{C \subset X \mid C \subset A \text{ and } C \text{ is open in } X\}.$$

Definition 1.8. The **boundary** of A, denoted by ∂A, is defined as:

$$\partial A = X \setminus (\operatorname{Int} A \cup \operatorname{Ext} A).$$

where

The **exterior** of a set A, denoted by $\operatorname{Ext} A$, is defined as:

$$\operatorname{Ext} A = X \setminus \overline{A}.$$

Definition 1.9. A map $f : X \to Y$ (continuous or not) is said to be an **open map** if it takes open subsets of X to open subsets of Y; in other words, if for every open subset $U \subseteq X$, the image set $f(U)$ is open in Y. It is said to be a **closed map** if it takes closed subsets of X to closed subsets of Y.

Definition 1.10. A topological space X is said to be a **Hausdorff space** if, given any pair of distinct points $p_1, p_2 \in X$, there exist neighborhoods U_1 of p_1 and U_2 of p_2 such that $U_1 \cap U_2 = \varnothing$.

Proposition 1.11. Any subspace A of a Hausdorff space is Hausdorff.

Proof. Let x and y be distinct points in A. Since $A \subseteq S$ and S is Hausdorff, there exist disjoint open neighborhoods U and V of x and y, respectively, in S. Then $U \cap A$ and $V \cap A$ are disjoint open neighborhoods of x and y in the subspace A. Hence, A is Hausdorff. $\square$

1.2 Bases and Countability

Definition 1.12. Let X be a topological space. A collection $\mathcal{B}$ of subsets of X is called a **basis** for the topology of X if the following two conditions hold:

1. Every element of $\mathcal{B}$ is an open subset of X.

2. Every open subset of X is the union of some collection of elements of $\mathcal{B}$.

Proposition 1.13. Let X and Y be topological spaces and let $\mathcal{B}$ be a basis for Y. A map $f : X \to Y$ is continuous if and only if for every basis subset $B \in \mathcal{B}$, the subset $f^{-1}(B)$ is open in X.

Proposition 1.14. Let X be a set, and suppose $\mathcal{B}$ is a collection of subsets of X. Then $\mathcal{B}$ is a basis for some topology on X if and only if it satisfies the following two conditions:

1. $\bigcup_{B \in \mathcal{B}} B = X$.
2. If $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, then there exists an element $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

If so, there is a unique topology on X for which $\mathcal{B}$ is a basis, called the **topology generated by** $\mathcal{B}$.

Definition 1.15. Let X be a topological space and $p \in X$. A collection $\mathcal{B}_p$ of neighborhoods of p is called a **neighborhood basis** for X at p if every neighborhood of p contains some $B \in \mathcal{B}_p$.We say that X is **first countable** if there exists a countable neighborhood basis at each point.

Definition 1.16. A topological space is said to be **second countable** if it admits a countable basis for its topology.

Proposition 1.17. [**Properties of Second Countable Spaces**] Suppose X is a second countable space.

1. X is first countable.
2. X contains a countable dense subset.
3. Every open cover of X has a countable subcover.

1.3 Inner Product Space

Definition 1.18. An *inner product* on a real vector space V is a map

$$\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}, \quad (v, w) \mapsto \langle v, w \rangle,$$

that satisfies the following properties for all $v, w, x \in V$ and $a, b \in \mathbb{R}$:

(i) Symmetry: $\langle v, w \rangle = \langle w, v \rangle$.

(ii) Bilinearity: $\langle av + bw, x\rangle = a\langle v, x\rangle + b\langle w, x\rangle = \langle x, av + bw\rangle$.

(iii) Positive Definiteness: $\langle v, v\rangle \geq 0$, with equality if and only if $v = 0$.

A vector space endowed with a specific inner product is called an **inner product space**.

For any $v \in V$, the *norm* or *length* of v is defined by

$$\|v\| = \sqrt{\langle v, v\rangle}.$$

Lemma 1.19. (Polarization Identity). Suppose $\langle \cdot, \cdot\rangle$ is an inner product on a vector space V. Then for all $v, w \in V$,

$$\langle v, w\rangle = \frac{1}{4}\left(\langle v + w, v + w\rangle - \langle v - w, v - w\rangle\right).$$

Proposition 1.20 (Gram–Schmidt Algorithm)**.** Let V be an n-dimensional inner product space, and suppose $(v_1, \ldots, v_n)$ is any ordered basis for V. Then there is an orthonormal ordered basis $(b_1, \ldots, b_n)$ satisfying the following condition:

$$\text{span}(\mathrm{b}_1, \ldots, b_k) = \text{span}(v_1, \ldots, v_k) \quad \text{for each } k = 1, \ldots, n.(1.1)$$

Proof. The basis vectors $b_1, \ldots, b_n$ are defined recursively by

$\mathrm{b}_1 = \frac{v_1}{\|v_1\|}, and for 2 \leq j \leq n,$

$\mathrm{b}_j = \frac{v_j - \sum_{i=1}^{j-1}\langle v_j, b_i\rangle b_i}{\left\|v_j - \sum_{i=1}^{j-1}\langle v_j, b_i\rangle b_i\right\|}.$

Because $v_1 \neq 0$ and $v_j \notin \text{span}(b_1, \ldots, b_{j-1})$ for each $j \geq 2$, the denominators are all nonzero.These vectors satisfy (1.1) by construction, and are orthonormal by direct computation.

□

Definition 1.21. If two vector spaces V and W are both equipped with inner products, denoted by $\langle \cdot, \cdot\rangle_V$ and $\langle \cdot, \cdot\rangle_W$, respectively, then a map $F \colon V \to W$ is called a **linear isometry** if it is a vector space isomorphism that preserves inner products:

$$\langle F(v), F(v')\rangle_W = \langle v, v'\rangle_V \quad \text{for all } v, v' \in V.$$

Chapter 2

TOPOLOGICAL MANIFOLDS

Topological manifolds generalize the familiar notions of curves and surfaces to higher dimensions. Informally, a topological manifold is a space that *locally* resembles Euclidean space. That is, each point lies in an open neighborhood that is homeomorphic to an open subset of $\mathbb{R}^n$ for some fixed n.To ensure that such spaces have good topological properties, additional conditions are imposed: the space must be *Hausdorff* and *second countable* Together, these properties define a **topological manifold**, which serves as a foundational concept in topology and geometry.
Let M be a topological space with the following properties:

- **Hausdorff:** For every pair of distinct points $p, q \in M$, there exist disjoint open subsets $U, V \subset M$ such that $p \in U$ and $q \in V$.
- **Second-countable:** There exists a countable basis for the topology of M.
- **Locally Euclidean of dimension n:** Each point of M has a neighborhood that is homeomorphic to an open subset of $\mathbb{R}^n$. More specifically, for each $p \in M$, we can find:
  - an open subset $U \subset M$ containing p,
  - an open subset $U' \subset \mathbb{R}^n$, and
  - a homeomorphism $\varphi : U \to U'$.

Example 2.1. [The Euclidean space $\mathbb{R}^n$]

The Euclidean space $\mathbb{R}^n$ is covered by a single chart $(\mathbb{R}^n, \mathrm{id}_{\mathbb{R}^n})$, where $\mathrm{id}_{\mathbb{R}^n} : \mathbb{R}^n \to \mathbb{R}^n$ is the identity map. It is the prime example of a topological manifold. Every open subset of $\mathbb{R}^n$ is also a topological manifold, with chart (U, id_U).

Recall that the Hausdorff condition and second countability are *hereditary properties*: that is, they are inherited by subspaces. A subspace of a Hausdorff space is Hausdorff and a subspace of a second-countable space is second-countable. So any subspace of $\mathbb{R}^n$ is automatically Hausdorff and second-countable.

Theorem 2.2. *Let M be a non-empty topological manifold of dimension n, and let N be a topological manifold of dimension m. If $M \cong N$ (i.e., they are homeomorphic), then $m = n$.*

Proof. Suppose, for contradiction, that M and N are homeomorphic topological manifolds with dimensions n and m, respectively, and that $m \neq n$. Without loss of generality, assume $m < n$. Let $f : M \to N$ be a homeomorphism.

Since M is an n-dimensional topological manifold, for any point $p \in M$, there exists an open neighborhood $U \subset M$ of p such that $U \cong \mathbb{R}^n$. The image $f(U)$ is then an open subset of N, and since f is a homeomorphism, we have

$$f(U) \cong U \cong \mathbb{R}^n.$$

However, since $f(U) \subset N$ and N is an m-dimensional manifold, there exists an open set $V \subset \mathbb{R}^m$ such that $f(U) \cong V$. Therefore, we have a homeomorphism

$$\mathbb{R}^n \cong f(U) \cong V \subset \mathbb{R}^m,$$

implying the existence of a homeomorphism between a non-empty open subset of $\mathbb{R}^n$ and one of $\mathbb{R}^m$. This contradicts the known result that $\mathbb{R}^n$ and $\mathbb{R}^m$ are not homeomorphic for $n \neq m$. Thus, our assumption must be false. Therefore, $m = n$. $\square$

2.1 Coordinate Charts

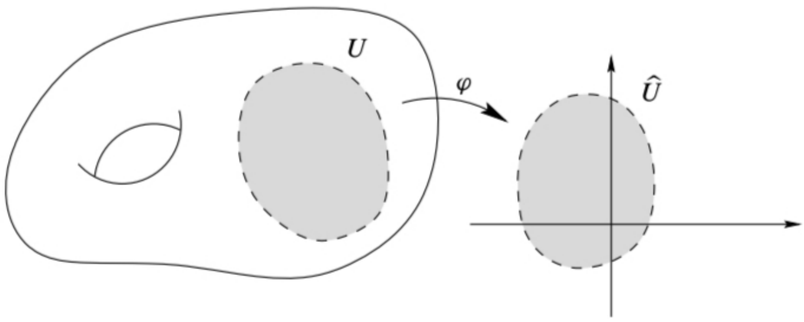

Figure 2.1: Coordinate Charts

Definition 2.3. Let M be a topological manifold of dimension n. A *coordinate chart* (or simply a *chart*) on M is a pair (U, φ), where:

- $U \subset M$ is an open subset (called the *domain* of the chart),
- $\varphi : U \to \varphi(U) \subset \mathbb{R}^n$ is a homeomorphism onto its image.

The map φ is called the *coordinate map*, and $\varphi(U)$ is an open subset of $\mathbb{R}^n$. The coordinates of a point $p \in U$ are given by the tuple $\varphi(p) \in \mathbb{R}^n$.

Theorem 2.4. Every topological manifold has a countable basis consisting of pre-compact coordinate balls.

Proof. Let M be a topological manifold. By definition, M is Hausdorff, second-countable, and locally Euclidean; that is, each point $p \in M$ has an open neighborhood U such that there exists a homeomorphism $\varphi : U \to \varphi(U) \subset \mathbb{R}^n$.

Since M is second-countable, there exists a countable basis $\{W_i\}_{i\in\mathbb{N}}$ for its topology. Moreover, since M is locally Euclidean and second-countable, it admits a countable atlas $\{(U_i, \varphi_i)\}_{i\in\mathbb{N}}$, where each $\varphi_i : U_i \to \varphi_i(U_i) \subset \mathbb{R}^n$ is a homeomorphism.

Fix a chart (U_i, φ_i). The image $\varphi_i(U_i)$ is an open subset of $\mathbb{R}^n$. For each point $x \in \varphi_i(U_i)$ and each rational radius $r > 0$ such that the closed Euclidean ball $B(x, r) \subset \varphi_i(U_i)$, define the set

$$B^{(i)}_{x,r} := \varphi_i^{-1}(B(x, r)) \subset M.$$

Each $B^{(i)}_{x,r}$ is an open subset of M homeomorphic to the Euclidean ball $B(x,r)$, and its closure in M lies within U_i. Since $B(x,r)$ is compact in $\mathbb{R}^n$ and φ_i is a homeomorphism, it follows that $B^{(i)}_{x,r}$ is compact and contained in U_i. Thus, $B^{(i)}_{x,r}$ is a *pre-compact coordinate ball.*

Define

$$\mathcal{B} := \left\{ B^{(i)}_{x,r} : i \in \mathbb{N},\ x \in \mathbb{Q}^n,\ r \in \mathbb{Q}_{>0},\ B(x,r) \subset \varphi_i(U_i) \right\}.$$

Since each chart domain $\varphi_i(U_i)$ is open in $\mathbb{R}^n$, and rational balls are countable in number, each collection $\{B^{(i)}_{x,r}\}$ is countable. Therefore, $\mathcal{B}$ is a countable union of countable sets, and hence countable.

We now show that $\mathcal{B}$ is a basis for the topology on M. Let $p \in M$, and let $V \subset M$ be an open neighborhood of p. Then p lies in some chart domain U_i, and hence $\varphi_i(p) \in \varphi_i(U_i)$. Since $\varphi_i(V \cap U_i)$ is an open neighborhood of $\varphi_i(p)$ in $\mathbb{R}^n$, there exists a rational ball $B(x,r) \subset \varphi_i(V \cap U_i)$ such that $B(x,r) \subset \varphi_i(U_i)$. Then $p \in B^{(i)}_{x,r} \subset V$, and $B^{(i)}_{x,r} \in \mathcal{B}$.

Therefore, $\mathcal{B}$ is a countable basis for the topology of M, and each element is a pre-compact coordinate ball. □

2.2 Connectivity of Manifolds

The existence of a basis of coordinate balls has important consequences for the connectivity properties of manifolds. Recall the following definitions for a topological space X:

- X is **connected** if there do not exist two disjoint, nonempty, open subsets of X whose union is X.
- X is **path-connected** if every pair of points in X can be joined by a path in X.
- X is **locally path-connected** if X has a basis of path-connected open subsets.

The following proposition shows that connectivity and path-connectedness coincide for manifolds.

Proposition 2.5. Let M be a topological manifold. Then:

(a) M is locally path-connected.

(b) M is connected if and only if it is path-connected.

(c) The components of M are the same as its path components.

(d) M has countably many components, each of which is an open subset of M and a connected topological manifold.

Proof. (a) Since M is a topological manifold, it has a basis of open sets homeomorphic to open subsets of $\mathbb{R}^n$, which are path-connected. Hence, M is locally path-connected.

(b) In locally path-connected spaces, connectedness implies path-connectedness and vice versa.

(c) In any locally path-connected space, the connected components coincide with the path components.

(d) Since M is second-countable, it has at most countably many components. Each component is open (because the space is locally path-connected) and inherits the manifold structure, hence is a connected topological manifold. □

Theorem 2.6. Let M be a topological manifold, let $\mathcal{U}$ be an open cover of M, and let $\mathcal{B}$ be a basis for the topology of M. Then there exists a countable, locally finite open refinement of $\mathcal{U}$ consisting of elements of $\mathcal{B}$. In particular, every topological manifold is paracompact.

Proof. Since M is a topological manifold, it is by definition Hausdorff, second-countable, and locally Euclidean.

Let $\mathcal{B} = \{B_i\}_{i\in\mathbb{N}}$ be a countable basis for the topology of M, which exists by second-countability. Let $\mathcal{U}$ be an arbitrary open cover of M.

Because M is locally Euclidean and Hausdorff, it is locally compact. It follows from standard results in general topology (specifically, the shrinking leproofor locally compact Hausdorff spaces) that there exists a locally finite open refinement $\{V_j\}_{j\in J}$ of $\mathcal{U}$ such that for each j, $V_j \subset U_j$ for some $U_j \in \mathcal{U}$.

Now, for each V_j and each $x \in V_j$, since $\mathcal{B}$ is a basis, there exists $B_{j,x} \in \mathcal{B}$ such that

$$x \in B_{j,x} \subset V_j.$$

Because V_j is open and $\mathcal{B}$ is a basis, we can find a countable subcollection $\{B_{j,k}\}_{k\in\mathbb{N}} \subset \mathcal{B}$ such that

$$V_j = \bigcup_{k\in\mathbb{N}} B_{j,k}, \quad \text{with } B_{j,k} \subset V_j.$$

Define

$$\mathcal{V} := \{B_{j,k} \mid j \in J, k \in \mathbb{N}\}.$$

Since $\mathcal{B}$ is countable and each V_j is covered by countably many basis elements, $\mathcal{V}$ is at most countable (being a countable union of countable sets). Moreover, each $B_{j,k} \subset V_j \subset U_j$, so $\mathcal{V}$ is a refinement of $\mathcal{U}$.

It remains to show that $\mathcal{V}$ is locally finite. Let $x \in M$. Since $\{V_j\}_{j\in J}$ is a locally finite cover, there exists an open neighborhood W of x such that W intersects only finitely many of the sets V_j, say for indices $j_1, \ldots, j_r$. Therefore, W can intersect only sets $B_{j,k}$ for $j \in \{j_1, \ldots, j_r\}$, each of which has countably many elements. But W is open and intersects only finitely many V_j, so it intersects only finitely many $B_{j,k}$.

Hence, $\mathcal{V}$ is locally finite. Thus, $\mathcal{V}$ is a countable, locally finite open refinement of $\mathcal{U}$ consisting of elements of the basis $\mathcal{B}$. This proves that M is paracompact. $\square$

2.3 Topological Manifolds with Boundary

An **n-dimensional topological manifold with boundary** is a second-countable Hausdorff space M such that every point $p \in M$ has a neighborhood homeomorphic either to an open subset of $\mathbb{R}^n$ or to an open subset (in the relative topology) of the closed upper half-space:

$$\mathbb{H}^n = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : x_n \geq 0\}.$$

A **chart** on M is a pair (U, φ), where $U \subseteq M$ is open and

$$\varphi : U \to \varphi(U) \subseteq \mathbb{R}^n \quad \text{or} \quad \varphi(U) \subseteq \mathbb{H}^n$$

is a homeomorphism onto an open subset of $\mathbb{R}^n$ or $\mathbb{H}^n$, respectively.

- An **interior chart** is a chart (U, φ) such that $\varphi(U)$ is an open subset of $\mathbb{R}^n$.

- A **boundary chart** is a chart (U, φ) such that $\varphi(U)$ is an open subset of $\mathbb{H}^n$ and $\varphi(U) \cap \partial\mathbb{H}^n \neq \emptyset$, where

$$\partial\mathbb{H}^n = \{x \in \mathbb{H}^n : x_n = 0\}.$$

- A **coordinate half-ball** is a boundary chart where

$$\varphi(U) = B_r(x) \cap \mathbb{H}^n$$

for some $x \in \partial\mathbb{H}^n$ and $r > 0$.

Interior and Boundary Points

Let $p \in M$.

- p is called an **interior point** of M if there exists an interior chart (U, φ) such that $p \in U$.
- p is called a **boundary point** of M if there exists a boundary chart (U, φ) such that $p \in U$ and $\varphi(p) \in \partial\mathbb{H}^n$.

We define the interior and boundary of M as:

$$\mathrm{Int}(M) = \{p \in M : p \text{ is an interior point}\}, \quad \partial M = \{p \in M : p \text{ is a boundary point}\}.$$

Remarks

1. Every point in M is either an interior point or a boundary point, but not both.
2. The classification of a point as interior or boundary is independent of the chart used; that is, a point cannot be mapped to $\mathbb{R}^n$ in one chart and to $\partial\mathbb{H}^n$ in another.
3. This follows from the compatibility of charts and the topological invariance of the boundary. Specifically, if a point is mapped to $\partial\mathbb{H}^n$ in one chart, it cannot be mapped to the interior of $\mathbb{R}^n$ under a homeomorphism (as such a homeomorphism would not preserve the boundary structure).

Theorem 2.7. (Topological Invariance of the Boundary)

Let M be a topological manifold with boundary. Then every point of M is either an interior point or a boundary point, but not both. That is,

$$M = \mathrm{Int}(M) \cup \partial M, \quad \mathrm{Int}(M) \cap \partial M = \emptyset.$$

Proof. Let $p \in M$ be arbitrary. Since M is a topological manifold with boundary, there exists a chart

$$(U, \varphi), \quad \varphi : U \to V \subset \mathbb{H}^n,$$

where $\mathbb{H}^n = \{x \in \mathbb{R}^n : x_n \geq 0\}$ is the closed upper half-space, and φ is a homeomorphism onto its image.

We consider two cases based on the image $\varphi(p) \in \mathbb{H}^n$:

Case 1: If $\varphi(p) \in \mathrm{Int}(\mathbb{H}^n)$, i.e., $(\varphi(p))_n > 0$, then there exists an open ball $B_\varepsilon(\varphi(p)) \subset V \subset \mathbb{H}^n$ entirely contained in the interior. Then $\varphi^{-1}(B_\varepsilon(\varphi(p))) \subset U$ is an open neighborhood of p homeomorphic to an open subset of $\mathbb{R}^n$. Hence, $p \in \mathrm{Int}(M)$.

Case 2: If $\varphi(p) \in \partial\mathbb{H}^n$, i.e., $(\varphi(p))_n = 0$, then by definition $p \in \partial M$.

Suppose, for contradiction, that $p \in \mathrm{Int}(M) \cap \partial M$. Then there exist two charts:

Suppose, for contradiction, that $p \in \mathrm{Int}(M) \cap \partial M$. Then there exist two charts:

- (U, φ) with $\varphi(p) \in \partial\mathbb{H}^n$,
- (W, ψ) with $\psi(p) \in \mathbb{R}^n$, i.e., ψ maps to an open subset of $\mathbb{R}^n$.

Consider the transition map:

$$\psi \circ \varphi^{-1} : \varphi(U \cap W) \to \psi(U \cap W).$$

This is a homeomorphism between an open subset of $\mathbb{H}^n$ (containing a boundary point) and an open subset of $\mathbb{R}^n$. But no open subset of $\mathbb{H}^n$ containing a boundary point is homeomorphic to an open subset of $\mathbb{R}^n$, since their local topological structures differ. This contradiction implies that no point can be both an interior and a boundary point.

Thus,

$$M = \mathrm{Int}(M) \cup \partial M, \qquad \mathrm{Int}(M) \cap \partial M = \emptyset.$$

□

Even though the term *manifold with boundary* encompasses manifolds without boundary as a special case, we will often use redundant phrases such as *manifold without boundary* when we wish to emphasize that we are talking about a manifold in the original sense. The phrase *manifold with or without boundary* will be used when results apply equally well to both kinds.

Note that the word “manifold” without further qualification always refers to a manifold without boundary.

Result :

Let M be a topological n-manifold with boundary. We analyze the following statements:

1. $\operatorname{Int} M$ **is an open subset of** M **and a topological** n**-manifold without boundary.**
   By definition, the interior $\operatorname{Int} M$ consists of all points in M that have neighborhoods homeomorphic to $\mathbb{R}^n$. These form an open subset of M, and locally resemble $\mathbb{R}^n$, making $\operatorname{Int} M$ an n-manifold without boundary.

2. ∂M **is a closed subset of** M **and a topological** $(n-1)$**-manifold without boundary.**
   The boundary ∂M consists of points in M with neighborhoods homeomorphic to open subsets of the upper half-space $\mathbb{H}^n$, where the boundary points map to $\mathbb{R}^{n-1} \subset \mathbb{H}^n$. Hence, ∂M is closed in M and locally resembles $\mathbb{R}^{n-1}$, so it is a topological $(n-1)$-manifold without boundary.

3. M **is a topological manifold if and only if** $\partial M = \emptyset$**.**
   A topological n-manifold is locally homeomorphic to $\mathbb{R}^n$. If $\partial M = \emptyset$, then all points are interior points, and M is a manifold without boundary. Conversely, if M is a manifold without boundary, then by definition $\partial M = \emptyset$.

4. **If** $n = 0$**, then** $\partial M = \emptyset$ **and** M **is a 0-manifold.**
   In dimension zero, $\mathbb{R}^0 = \{0\}$ and $\mathbb{H}^0 = \{0\}$ are the same set, so there is no meaningful notion of boundary. Every point is both an interior and boundary point, so we define $\partial M = \emptyset$. Therefore, M is simply a discrete set of points — a 0-manifold.

Chapter 3

SMOOTH MANIFOLDS

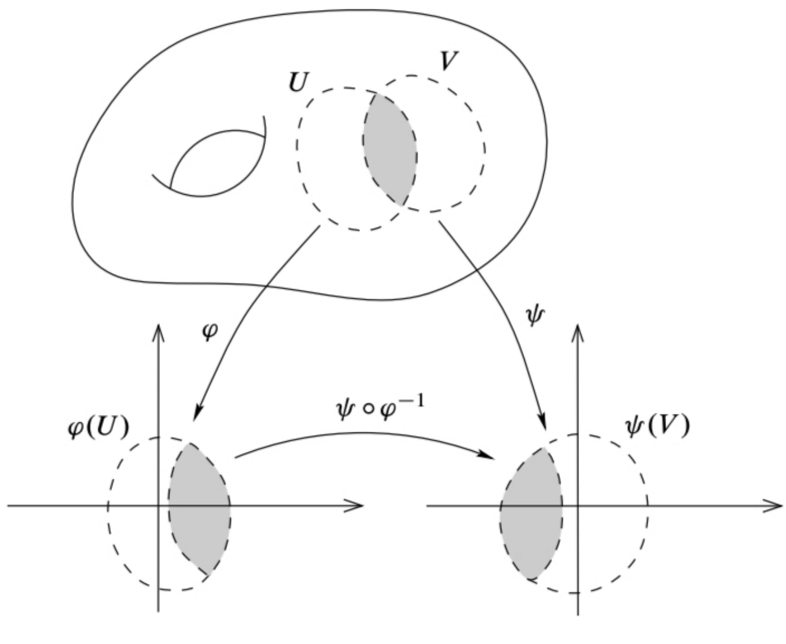

Figure 3.1: Transition map

Let M be a topological n-manifold. If (U, φ) and (V, ψ) are two charts such that $U \cap V \neq \emptyset$, the composite map

$$\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$$

is called the **transition map** from φ to ψ. It is a composition of homeomorphisms, and is therefore itself a homeomorphism.

Two charts (U, φ) and (V, ψ) are said to be **smoothly compatible** if either $U \cap V = \emptyset$ or the transition map $\psi \circ \varphi^{-1}$ is a **diffeomorphism**. Since $\varphi(U \cap V)$ and $\psi(U \cap V)$ are open subsets of $\mathbb{R}^n$, smoothness of this map is to be interpreted in the ordinary sense: having continuous partial derivatives of all orders.

An **atlas** for M is a collection of charts whose domains cover M. An atlas $\mathcal{A}$ is called a **smooth atlas** if any two charts in $\mathcal{A}$ are smoothly compatible with each other.

To make sense of derivatives of real-valued functions, curves, or maps between manifolds, we need to introduce a new kind of manifold called a **smooth manifold**. It is a topological manifold equipped with some extra structure in addition to its topology, allowing us to determine which functions to or from the manifold are smooth.

The definition is based on the calculus of maps between Euclidean spaces. So we recall:

Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ be open subsets. A function

$$F : U \to V$$

is said to be **smooth** (or C^∞, or infinitely differentiable) if each of its component functions has continuous partial derivatives of all orders. If in addition F is bijective and has a smooth inverse map, it is called a **diffeomorphism**. A diffeomorphism is, in particular, a homeomorphism.

A smooth atlas $\mathcal{A}$ on M is **maximal** if it is not properly contained in any larger smooth atlas. This means that any chart that is smoothly compatible with every chart in $\mathcal{A}$ is already in $\mathcal{A}$. (Such a smooth atlas is also called **complete**.)

Smooth Manifold

Let M be a topological manifold. A **smooth structure** on M is a maximal smooth atlas. A **smooth manifold** is a pair $(M, \mathcal{A})$, where M is a topological manifold and $\mathcal{A}$ is a smooth structure on M.
When the smooth structure is understood, we usually omit mention of it and just say "M is a smooth manifold." Smooth structures are also called **differentiable structures** or C^∞ **structures** by some authors. We also use the term **smooth manifold structure** to mean a manifold topology together with a smooth structure.

If M is a smooth manifold, any chart (U, φ) contained in the given maximal smooth atlas is called a **smooth chart**, and the corresponding map φ is called a **smooth coordinate map**. The set U is often referred to as a **smooth coordinate domain** or **smooth coordinate neighborhood**.

- A **smooth coordinate ball** is a smooth coordinate domain whose image under a smooth coordinate map is a ball in Euclidean space.
- A **smooth coordinate cube** is defined similarly, with the image being a cube in $\mathbb{R}^n$.

It is often useful to restrict attention to coordinate balls whose closures sit nicely inside larger coordinate balls. A set $B \subset M$ is called a **regular coordinate ball** if there exists a smooth coordinate ball $B_0 \supset B$ and a smooth coordinate map $\varphi : B_0 \to \mathbb{R}^n$ such that for some positive real numbers $r < r_0$,

$$\varphi(B) = B_r(0), \quad B \subset B_x = \varphi^{-1}(B_r(0)), \quad \text{and} \quad B_0 = \varphi^{-1}(B_{r_0}(0)).$$

Since B_x is homeomorphic to $B_r(0)$, it is compact, and thus every regular coordinate ball is precompact in M.

Example 3.1. (Graphs of Continuous Functions) Let $U \subset \mathbb{R}^n$ be an open subset, and let $f : U \to \mathbb{R}^k$ be a continuous function. The graph of f is the subset of $\mathbb{R}^n \times \mathbb{R}^k$ defined by

$$\Gamma(f) = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^k \mid x \in U \text{ and } y = f(x)\},$$

with the subspace topology.
Let $\pi_1 : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n$ denote the projection onto the first factor, and let $\varphi : \Gamma(f) \to U$ be the restriction of π_1 to $\Gamma(f)$:

$$\varphi(x, y) = x, \quad \text{for } (x, y) \in \Gamma(f).$$

Because φ is the restriction of a continuous map, it is continuous; and it is a homeomorphism because it has a continuous inverse given by $\varphi^{-1}(x) = (x, f(x))$. Thus, $\Gamma(f)$ is a topological manifold of dimension n. In fact, $\Gamma(f)$ is homeomorphic to U itself, and $(\Gamma(f), \varphi)$ is a global coordinate chart, called *graph coordinates*.
The same observation applies to any subset of $\mathbb{R}^{n+k}$ defined by setting any k of the coordinates (not necessarily the last k) equal to some continuous function of the other n, which are restricted to lie in an open subset of $\mathbb{R}^n$.

Example 3.2. (Spheres) For each integer $n \geq 0$, the unit n-sphere S^n is Hausdorff and second-countable because it is a topological subspace of $\mathbb{R}^{n+1}$. To show that it is locally Euclidean, for each index $i = 1, \ldots, n+1$, let U_i^+ denote the subset of $\mathbb{R}^{n+1}$ where the i-th coordinate is positive:

$$U_i^+ = \left\{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_i > 0\right\}.$$

Similarly, define U_i^- as the set where $x_i < 0$.
Let $f : B^n \to \mathbb{R}$ be the continuous function

$$f(u) = \sqrt{1 - \|u\|^2}.$$

Then for each $i = 1, \ldots, n+1$, it is easy to check that $U_i^+ \cap S^n$ is the graph of the function

$$x_i = f(x_1, \ldots, \hat{x}_i, \ldots, x_{n+1}),$$

where the hat indicates that x_i is omitted. Similarly, $U_i^- \cap S^n$ is the graph of

$$x_i = -f(x_1, \ldots, \hat{x}_i, \ldots, x_{n+1}).$$

Thus, each subset $U_i^\pm \cap S^n$ is locally Euclidean of dimension n, and the maps

$$\varphi_i^\pm : U_i^\pm \cap S^n \to B^n$$

are given by

$$\varphi_i^\pm(x_1, \ldots, x_{n+1}) = (x_1, \ldots, \hat{x}_i, \ldots, x_{n+1}),$$

are graph coordinates of S^n .Since each point of S^n is in the domain of at least one of these $2n+2$ charts, S^n is a topological n-manifold.

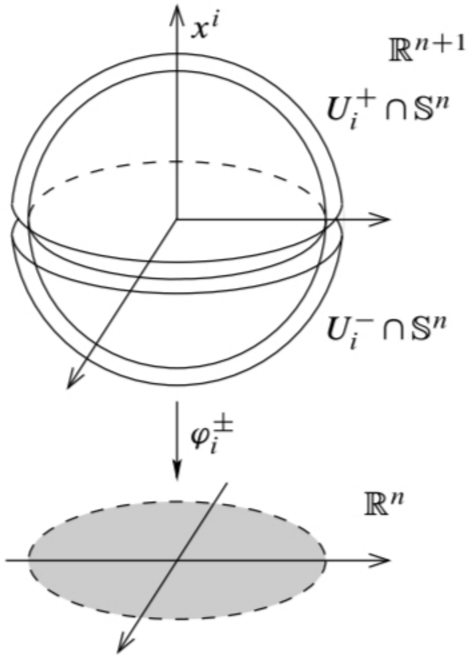

Figure 3.2: Charts for S^n

Example 3.3. (Projective Spaces) The n-dimensional real projective space, denoted by $\mathbb{RP}^n$ (or sometimes just $\mathbb{P}^n$), is defined as the set of 1-dimensional linear subspaces of $\mathbb{R}^{n+1}$, with the quotient topology determined by the natural map

$$q : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{RP}^n$$

that sends each point $x \in \mathbb{R}^{n+1} \setminus \{0\}$ to the subspace spanned by x. The 2-dimensional projective space $\mathbb{RP}^2$ is called the *projective plane*. For any point $x \in \mathbb{R}^{n+1} \setminus \{0\}$, let $[x] = q(x) \in \mathbb{RP}^n$ denote the line spanned by x. For each $i = 1, \ldots, n+1$, let

$$\tilde{U}_i = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \setminus \{0\} \mid x_i \neq 0\},$$

and let $U_i = q(\tilde{U}_i) \subset \mathbb{RP}^n$. Since $\tilde{U}_i$ is a saturated open subset, U_i is open, and the restriction $q|_{\tilde{U}_i} : \tilde{U}_i \to U_i$ is a quotient map
Define a map $\varphi_i : U_i \to \mathbb{R}^n$ by

$$\varphi_i([x_1, \ldots, x_{n+1}]) = \left(\frac{x_1}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_{n+1}}{x_i}\right).$$

This map is well defined because its value is unchanged by multiplying x by a nonzero constant. Since $\varphi_i \circ q$ is continuous, φ_i is continuous by the characteristic property of quotient maps
In fact, φ_i is a homeomorphism, because it has a continuous inverse given by

$$\varphi_i^{-1}(u_1, \ldots, u_n) = [u_1, \ldots, u_{i-1}, 1, u_i, \ldots, u_n],$$

Geometrically, $\varphi_i([x]) = u$ means that $(u, 1)$ is the point in $\mathbb{R}^{n+1}$ where the line $[x]$ intersects the affine hyperplane where $x_i = 1$ Because the sets $U_1, \ldots, U_{n+1}$ cover $\mathbb{RP}^n$, this shows that $\mathbb{RP}^n$ is locally Euclidean of dimension n.

Example 3.4. (Product Manifolds) Suppose $M_1, \ldots, M_k$ are topological manifolds of dimensions $n_1, \ldots, n_k$, respectively. The product space

$$M_1 \times \cdots \times M_k$$

is a topological manifold of dimension $n_1 + \cdots + n_k$ as follows. It is Hausdorff and second-countable , so only the locally Euclidean property needs to be checked.
Given any point $(p_1, \ldots, p_k) \in M_1 \times \cdots \times M_k$, we can choose a coordinate chart (U_i, φ_i) for each M_i with $p_i \in U_i$. The product map

$$\varphi_1 \times \cdots \times \varphi_k : U_1 \times \cdots \times U_k \to \mathbb{R}^{n_1+\cdots+n_k}$$

is a homeomorphism onto its image, which is a product open subset of $\mathbb{R}^{n_1+\cdots+n_k}$. Thus, $M_1 \times \cdots \times M_k$ is a topological manifold of dimension $n_1 + \cdots + n_k$, with charts of the form

$$(U_1 \times \cdots \times U_k, \varphi_1 \times \cdots \times \varphi_k).$$

Example 3.5. (Tori) For a positive integer n, the *n-torus* (plural: tori) is the product space

$$\mathbb{T}^n = S^1 \times \cdots \times S^1 \quad (n \text{ factors}).$$

By the discussion above, it is a topological n-manifold. (The 2-torus is usually called simply *the torus*.)

Definition 3.6. Let M be a smooth manifold. A coordinate chart (U, φ) is called *regular* if $\varphi(U)$ is an open ball in $\mathbb{R}^n$, and $\varphi : U \to \varphi(U)$ is a diffeomorphism.

A **regular coordinate ball** is an open subset $U \subset M$ such that there exists a chart (U, φ) with $\varphi(U) = B_r(x) \subset \mathbb{R}^n$, the open ball of radius r centered at x.

Proposition 3.7. *Every smooth manifold has a countable basis consisting of regular coordinate balls.*

3.1 Smooth Functions on Manifolds

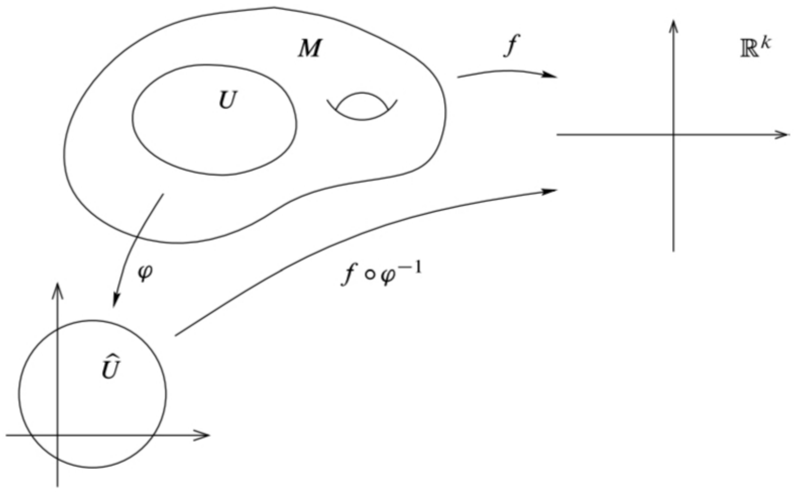

Figure 3.3: Definition of Smooth functions

Suppose M is a smooth n-manifold, k is a nonnegative integer, and $f : M \to \mathbb{R}^k$ is any function. We say that f is a *smooth function* if for every $p \in M$, there exists a smooth chart (U, φ) for M whose domain contains p, and such that the composite function

$$f \circ \varphi^{-1}$$

is smooth on the open subset $\varphi(U) \subseteq \mathbb{R}^n$.
If M is a smooth manifold with boundary, the definition is exactly the same, except that $\varphi(U)$ is now an open subset of either $\mathbb{R}^n$ or the closed half-space $\mathbb{H}^n$, and in the latter case, we interpret the smoothness of $f \circ \varphi^{-1}$ to mean that each point of $\varphi(U)$ has a neighborhood (in $\mathbb{R}^n$) on which $f \circ \varphi^{-1}$ extends to a smooth function in the ordinary sense.

3.2 Smooth Maps Between Manifolds

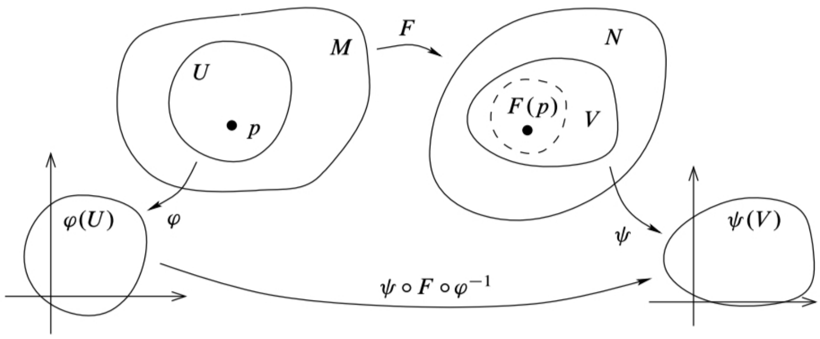

Figure 3.4: Definition of Smooth maps

The definition of smooth functions generalizes easily to maps between manifolds.
Let M, N be smooth manifolds, and let $F : M \to N$ be any map. We say that F is a *smooth map* if for every $p \in M$, there exist smooth charts (U, φ) containing p and (V, ψ) containing $F(p)$ such that $F(U) \subseteq V$ and the composite map

$$\psi \circ F \circ \varphi^{-1}$$

is smooth from $\varphi(U)$ to $\psi(V)$.
If M and N are smooth manifolds with boundary, smoothness of F is defined in exactly the same way, with the usual understanding that a map whose domain is a subset of $\mathbb{H}^n$ is smooth if it admits an extension to a smooth map in a neighborhood of each point and a map whose codomain is a subset of $\mathbb{H}^n$ is smooth if it is smooth as a map into $\mathbb{R}^n$.

Proposition 3.8. Every smooth map is continuous.

Proof. Suppose M and N are smooth manifolds, with or without boundary, and $F : M \to N$ is smooth. Given $p \in M$, smoothness of F means there are smooth charts (U, φ) containing p and (V, ψ) containing $F(p)$, such that $F(U) \subseteq V$ and the map

$$\psi \circ F \circ \varphi^{-1} : \varphi(U) \to \psi(V)$$

is smooth, hence continuous.
Since $\varphi : U \to \varphi(U)$ and $\psi : V \to \psi(V)$ are homeomorphisms, this implies in turn that

$$F|_U = \psi^{-1} \circ (\psi \circ F \circ \varphi^{-1}) \circ \varphi : U \to V$$

is continuous. Therefore, F is continuous at every point $p \in M$, and hence continuous on M. $\square$

3.3 Submanifolds

Many familiar manifolds appear naturally as subsets of other manifolds. We have already seen that open subsets of smooth manifolds can be viewed as smooth manifolds in their own right; but there are many interesting examples beyond the open ones.
In this chapter, we explore **smooth submanifolds**, which are smooth manifolds that are subsets of other smooth manifolds. As you will soon discover, the situation is quite a bit more subtle than the analogous theory of topological subspaces.
We begin by defining the most important type of smooth submanifolds, called **embedded submanifolds**. These have the subspace topology inherited from their containing manifold and turn out to be exactly the images of smooth embeddings. As we will see in this chapter, they are modeled locally on linear subspaces of Euclidean spaces.
Because embedded submanifolds are most often presented as level sets of smooth maps, we devote some time to analyzing the conditions under which level sets are smooth submanifolds. We will see, for example, that level sets of constant-rank maps (in particular, smooth submersions) are always embedded submanifolds.
Next, we introduce a more general kind of submanifold, called **immersed submanifolds**, which turn out to be the images of injective immersions. An immersed submanifold looks locally like an embedded one, but globally it may have a topology that is different from the subspace topology.

Definition 3.9. [Embedding Submanifold]
Suppose M is a smooth manifold (with or without boundary). An **embedded submanifold** of M is a subset $S \subseteq M$ that is a manifold (without boundary) in the subspace topology, endowed with a smooth structure with respect to which the inclusion map $S \hookrightarrow M$ is a smooth

embedding. Embedded submanifolds are also called **regular submanifolds** by some authors.
If S is an embedded submanifold of M, the difference $\dim M - \dim S$ is called the **codimension** of S in M, and the manifold M is called the **ambient manifold** for S. An **embedded hypersurface** is an embedded submanifold of codimension 1. The empty set $\emptyset$ is an embedded submanifold of any dimension.
The easiest embedded submanifolds to understand are those of codimension 0. For any smooth manifold M, we defined an open submanifold of M to be any open subset with the subspace topology and with the smooth charts obtained by restricting those of M.
Many examples of embedded submanifolds arise naturally as subsets of Euclidean spaces. A simple example is the unit circle

$$S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\},$$

which is a 1-dimensional embedded submanifold of $\mathbb{R}^2$. Similarly, the unit sphere

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$$

is a 2-dimensional embedded submanifold of $\mathbb{R}^3$. Another example is a straight line in the plane:

$$\{(x, y) \in \mathbb{R}^2 : y = mx + c\},$$

which is a 1-dimensional submanifold of $\mathbb{R}^2$.
More complex examples include the torus $T^2 \subset \mathbb{R}^3$, which can be realized as a surface of revolution and forms a 2-dimensional embedded submanifold of 3-space. The graph of a smooth function $f : \mathbb{R}^n \to \mathbb{R}^m$, given by

$$\{(x, f(x)) \in \mathbb{R}^{n+m} : x \in \mathbb{R}^n\},$$

is an n-dimensional embedded submanifold of $\mathbb{R}^{n+m}$.
Additionally, some more abstract manifolds like the real projective plane $\mathbb{RP}^2$ can be embedded into higher-dimensional Euclidean spaces, such as $\mathbb{R}^5$. Even non-orientable surfaces like the Möbius strip can be realized as embedded 2-dimensional submanifolds in $\mathbb{R}^3$.
These examples illustrate how diverse and geometrically rich embedded submanifolds can be.

Proposition 3.10. (Open Submanifolds)
Suppose M is a smooth manifold. The embedded submanifolds of codimension 0 in M are exactly the open submanifolds.

Definition 3.11. [Immersed Submanifolds]
Although embedded submanifolds are the most natural and common submanifolds and suffice for most purposes, it is sometimes important to consider a more general notion of submanifold.Let M be a smooth manifold with or without boundary. An **immersed submanifold** of M is a subset $S \subseteq M$ endowed with a topology (not necessarily the subspace topology) such that S is a topological manifold (without boundary), and a smooth structure with respect to which the inclusion map $S \hookrightarrow M$ is a smooth immersion. As with embedded submanifolds, we define the **codimension** of S in M to be $\dim M - \dim S$.
Every embedded submanifold is also an immersed submanifold. Because immersed submanifolds are more general, we adopt the convention that the term *smooth submanifold*, without further qualification, means an immersed one, which includes an embedded submanifold as a special case. Similarly, the term *smooth hypersurface* without qualification means an immersed submanifold of codimension 1.
There are also various notions of submanifolds in the topological category. For example, if M is a topological manifold, one could define an **immersed topological submanifold** of M to be a subset $S \subseteq M$ endowed with a topology such that S is a topological manifold and the inclusion map is a topological immersion. It is an **embedded topological submanifold** if the inclusion is a topological embedding.

Chapter 4

TANGENT VECTORS

Given a point $a \in \mathbb{R}^n$, we define the *geometric tangent space* to $\mathbb{R}^n$ at a, denoted by $\mathbb{R}^n_a$, to be the set

$$\mathbb{R}^n_a = \{(a, v) \mid v \in \mathbb{R}^n\} \subset \mathbb{R}^n \times \mathbb{R}^n.$$

A **geometric tangent vector** in $\mathbb{R}^n$ is an element of $\mathbb{R}^n_a$ for some $a \in \mathbb{R}^n$. As a matter of notation, we abbreviate (a, v) as v_a (or sometimes $v|_a$ if it is clearer—for example, if v itself has a subscript). We think of v_a as the vector v with its initial point at a .
The set $\mathbb{R}^n_a$ is a real vector space under the natural operations:

$$v_a + w_a = (v + w)_a, \quad c \cdot v_a = (cv)_a.$$

The vectors $e_i|_a$, for $i = 1, \ldots, n$, form a basis for $\mathbb{R}^n_a$. In fact, as a vector space, $\mathbb{R}^n_a$ is essentially the same as $\mathbb{R}^n$ itself; the only reason we add the index a is so that the geometric tangent spaces $\mathbb{R}^n_a$ and $\mathbb{R}^n_b$ at distinct points a and b will be disjoint sets.
With this definition, we could think of the tangent space to S^{n-1} at a point $a \in S^{n-1}$ as a certain subspace of $\mathbb{R}^n_a$ namely the space of vectors that are orthogonal to the radial unit vector through a, using the inner product that $\mathbb{R}^n_a$ inherits from $\mathbb{R}^n$ via the natural isomorphism $\mathbb{R}^n \cong \mathbb{R}^n_a$.

Lemma 4.1. (Properties of Derivations) Suppose $a \in \mathbb{R}^n$, $w \in T_a\mathbb{R}^n$, and $f, g \in C^\infty(\mathbb{R}^n)$.

(a) If f is a constant function, then $w(f) = 0$.

(b) If $f(a) = g(a) = 0$, then $w(fg) = 0$.

Proof. It suffices to prove (a) for the constant function $f_1(x) \equiv 1$, for then $f(x) \equiv c$ implies

$$w(f) = w(cf_1) = c \cdot w(f_1) = 0$$

by linearity. For f_1, the product rule gives

$$w(f_1) = w(f_1 \cdot f_1) = f_1(a) \cdot w(f_1) + f_1(a) \cdot w(f_1) = 2w(f_1),$$

which implies that $w(f_1) = 0$.
Similarly, (b) follows from the product rule:

$$w(fg) = f(a) \cdot w(g) + g(a) \cdot w(f) = 0 + 0 = 0.$$

□

4.1 Tangent Vectors on Manifolds

Now we are in a position to define tangent vectors on manifolds and manifolds with boundary. The definition is the same in both cases.
Let M be a smooth manifold with or without boundary, and let p be a point of M. A linear map $v : C^\infty(M) \to \mathbb{R}$ is called a *derivation at p* if it satisfies

$$v(fg) = f(p) \cdot v(g) + g(p) \cdot v(f)$$

The set of all derivations of $C^\infty(M)$ at p, denoted by T_pM, is a vector space called the *tangent space* to M at p. An element of T_pM is called a *tangent vector at p*.

Lemma 4.2. (Properties of Tangent Vectors on Manifolds) Suppose M is a smooth manifold with or without boundary, $p \in M$, $v \in T_pM$, and $f, g \in C^\infty(M)$.

(a) If f is a constant function, then $v(f) = 0$.

(b) If $f(p) = g(p) = 0$, then $v(fg) = 0$.

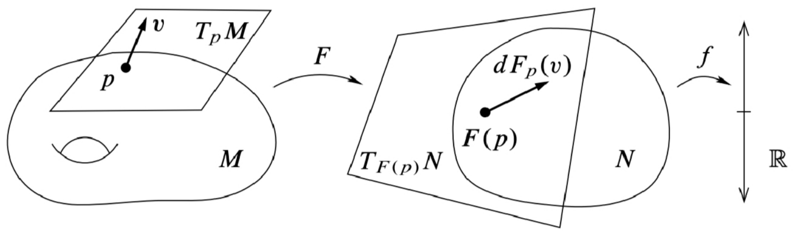

Figure 4.1: The differential

4.2 The Differential of a Smooth Map

If M and N are smooth manifolds (with or without boundary), and $F : M \to N$ is a smooth map, then for each point $p \in M$, we define a map

$$dF_p : T_pM \to T_{F(p)}N,$$

called the *differential* of F at p (see Fig. 4.1), as follows.
Given $v \in T_pM$, we define $dF_p(v)$ to be the derivation at $F(p)$ that acts on $f \in C^1(N)$ by the rule

$$dF_p(v)(f) := v(f \circ F).$$

Note that if $f \in C^1(N)$, then $f \circ F \in C^1(M)$, so the expression $v(f \circ F)$ makes sense. The operator $dF_p(v) : C^1(N) \to \mathbb{R}$ is linear because v is linear, and it is a derivation at $F(p)$ because, for any $f, g \in C^1(N)$, we have:

$$\begin{aligned} dF_p(v)(fg) &= v\left((fg) \circ F\right) \\ &= v\left((f \circ F)(g \circ F)\right) \\ &= f(F(p)) \cdot v(g \circ F) + g(F(p)) \cdot v(f \circ F) \\ &= f(F(p)) \cdot dF_p(v)(g) + g(F(p)) \cdot dF_p(v)(f), \end{aligned}$$

which shows that $dF_p(v)$ satisfies the Leibniz rule and is therefore a derivation at $F(p)$.

Proposition 4.3. (Properties of Differentials)
Let M, N, and P be smooth manifolds with or without boundary. Let $F : M \to N$ and $G : N \to P$ be smooth maps, and let $p \in M$. Then:

(a) The map $dF_p : T_pM \to T_{F(p)}N$ is linear.

(b) The differential satisfies the chain rule:

$$d(G \circ F)_p = dG_{F(p)} \circ dF_p : T_pM \to T_{G(F(p))}P.$$

(c) The differential of the identity map is the identity:

$$d(\mathrm{Id}_M)_p = \mathrm{Id}_{T_pM} : T_pM \to T_pM.$$

(d) If F is a diffeomorphism, then $dF_p : T_pM \to T_{F(p)}N$ is an isomorphism, and

$$(dF_p)^{-1} = d(F^{-1})_{F(p)}.$$

Proposition 4.4. Let M be a smooth manifold with or without boundary, $p \in M$, and $v \in T_pM$. If $f, g \in C^\infty(M)$ agree on some neighborhood of p, then

$$vf = vg.$$

Proposition 4.5. (The Tangent Space to an Open Submanifold)
Let M be a smooth manifold with or without boundary, let $U \subset M$ be an open subset, and let $\iota : U \hookrightarrow M$ be the inclusion map. For every $p \in U$, the differential

$$d\iota_p : T_pU \to T_pM$$

is an isomorphism.

Proof. To prove injectivity,Suppose $v \in T_pU$ and $d\iota_p(v) = 0 \in T_pM$. Let B be a neighborhood of p such that $B \subset U$. If $f \in C^\infty(U)$ is arbitrary, then by the smooth function extension lemma, there exists $\tilde{f} \in C^\infty(M)$ such that $\tilde{f} = f$ on B.
Since f and $\tilde{f}|_U$ agree in a neighborhood of p, proposition 4.4 implies:

$$vf = v(\tilde{f}|_U) = v(\tilde{f} \circ \iota) = d\iota_p(v)\tilde{f} = 0.$$

This holds for all $f \in C^\infty(U)$, so $v = 0$. Hence, $d\iota_p$ is injective.
To prove surjectivity,Let $w \in T_pM$ be arbitrary. Define an operator $v : C^\infty(U) \to \mathbb{R}$ by choosing, for each $f \in C^\infty(U)$, a smooth extension $\tilde{f} \in C^\infty(M)$ such that $\tilde{f} = f$ on a neighborhood B of p, and set:

$$vf := w\tilde{f}.$$

By proposition 4.4, this definition is independent of the choice of $\tilde{f}$, so v is well-defined. It is straightforward to check that v is a derivation at p in U, i.e., $v \in T_pU$.
For any $g \in C^\infty(M)$:

$$d\iota_p(v)(g) = v(g \circ \iota) = w\tilde{g} = wg,$$

since $g \circ \iota$ and $\tilde{g}$ agree near p. Therefore, $d\iota_p(v) = w$, so $d\iota_p$ is surjective.

□

4.3 The Differential in Coordinates

We explore how differentials appear in coordinates, beginning with the special case of a smooth map

$$F : U \to V,$$

where $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ are open subsets of Euclidean spaces.
For any point $p \in U$, we determine the matrix of the differential

$$dF_p : T_p\mathbb{R}^n \to T_{F(p)}\mathbb{R}^m$$

in terms of standard coordinate bases.
Let $x^1, \ldots, x^n$ denote the coordinates in the domain U, and let $y^1, \ldots, y^m$ denote the coordinates in the codomain V.
Using the chain rule, we compute the action of dF_p on a basis vector as follows:

$$dF_p\left(\left.\frac{\partial}{\partial x^i}\right|_p\right)(f) = \left.\frac{\partial}{\partial x^i}\right|_p (f \circ F) = \left.\frac{\partial f}{\partial y^j}\right|_{F(p)} \cdot \left.\frac{\partial F^j}{\partial x^i}\right|_p = \left(\left.\frac{\partial F^j}{\partial x^i}\right|_p \cdot \left.\frac{\partial}{\partial y^j}\right|_{F(p)}\right)(f).$$

Thus,

$$dF_p\left(\left.\frac{\partial}{\partial x^i}\right|_p\right) = \left.\frac{\partial F^j}{\partial x^i}\right|_p \left.\frac{\partial}{\partial y^j}\right|_{F(p)}.$$

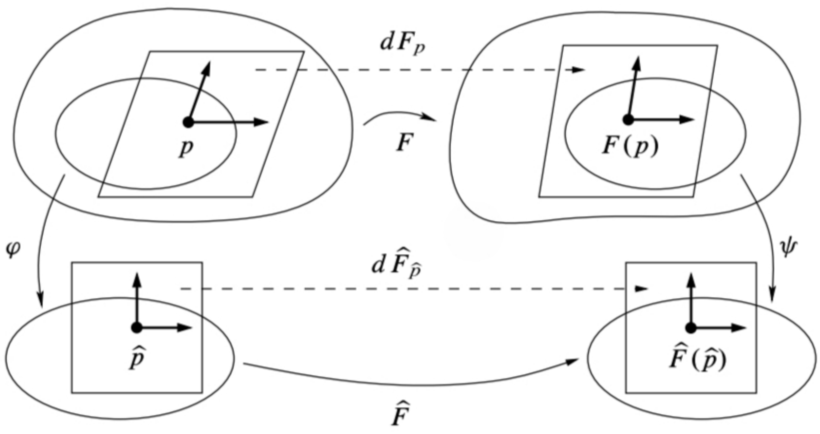

Figure 4.2: The differential in coordinates

In other words, the matrix of dF_p in terms of the coordinate bases is

$$\begin{bmatrix} \left.\frac{\partial F^1}{\partial x^1}\right|_p & \cdots & \left.\frac{\partial F^1}{\partial x^n}\right|_p \\ \vdots & \ddots & \vdots \\ \left.\frac{\partial F^m}{\partial x^1}\right|_p & \cdots & \left.\frac{\partial F^m}{\partial x^n}\right|_p \end{bmatrix}.$$

(Recall that the columns of the matrix are the components of the images of the basis vectors.) This matrix is none other than the Jacobian matrix of F at p, which is the matrix representation of the total derivative

$$DF(p)\colon \mathbb{R}^n \to \mathbb{R}^m.$$

Therefore, in this case, $dF_p\colon T_p\mathbb{R}^n \to T_{F(p)}\mathbb{R}^m$ corresponds to the total derivative $DF(p)\colon \mathbb{R}^n \to \mathbb{R}^m$, under our usual identification of Euclidean spaces with their tangent spaces. The same calculation applies if $U \subseteq \mathbb{H}^n$ and $V \subseteq \mathbb{H}^m$ are open subsets of the upper half-spaces.
Now consider the more general case of a smooth map $F\colon M \to N$ between smooth manifolds with or without boundary. Choosing smooth coordinate charts (U, φ) for M containing p, and (V, ψ) for N containing $F(p)$, we obtain the coordinate representation
$F_y = \psi \circ F \circ \varphi^{-1}\colon \varphi(U \cap F^{-1}(V)) \to \psi(V)$ (see Fig. 4.2). Let $p_y = \varphi(p)$ denote the coordinate representation of p. By the computation above,

$dF_y|_{p_y}$ is represented with respect to the standard coordinate bases by the Jacobian matrix of F_y at p_y. Using the fact that $F \circ \varphi^{-1} = \psi^{-1} \circ F_y$, we compute:

$$dF_p\left(\left.\frac{\partial}{\partial x^i}\right|_p\right) = dF_p\left(d\varphi^{-1}_{p_y}\left(\left.\frac{\partial}{\partial x^i}\right|_{p_y}\right)\right) = d\psi^{-1}_{F_y(p_y)}\left(dF_y|_{p_y}\left(\left.\frac{\partial}{\partial x^i}\right|_{p_y}\right)\right).$$

$$= d\psi^{-1}_{F_y(p_y)}\left(dF_y|_{p_y}\left(\left.\frac{\partial}{\partial x^i}\right|_{p_y}\right)\right) = \left.\frac{\partial F_y^j}{\partial x^i}\right|_{p_y}\left.\frac{\partial}{\partial y^j}\right|_{F(p)}.$$

Thus, dF_p is represented in coordinate bases by the Jacobian matrix of the coordinate representation F_y of F.

4.4 Change of Coordinates

Suppose (U, φ) and (V, ψ) are two smooth charts on M, and let $p \in U \cap V$. Let the coordinate functions of φ be denoted by x^i, and those of ψ by z^i. Any tangent vector at p can be represented with respect to either basis

$$\left.\frac{\partial}{\partial x^i}\right|_p \quad \text{or} \quad \left.\frac{\partial}{\partial z^i}\right|_p.$$

In this situation, it is customary to write the transition map $\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$ in the shorthand notation:

$$\psi \circ \varphi^{-1}(x) = \left(z^1(x), \ldots, z^n(x)\right).$$

Here, we indulge in a typical abuse of notation: in the expression $\tilde{x}^i(x), \tilde{x}^i$ as a coordinate function (whose domain is an open subset of M, identified with an open subset of $\mathbb{R}^n$ or $\mathbb{H}^n$); and we think of x as representing a point in $\varphi(U \cap V)$,the differential of $\psi \circ \varphi^{-1}$ at $\varphi(p)$ can be written as

$$d(\psi \circ \varphi^{-1})_{\varphi(p)}\left(\left.\frac{\partial}{\partial x^i}\right|_{\varphi(p)}\right) = \left.\frac{\partial \tilde{x}^j}{\partial x^i}\right|_{\varphi(p)}\left.\frac{\partial}{\partial \tilde{x}^j}\right|_{\psi(p)}.$$

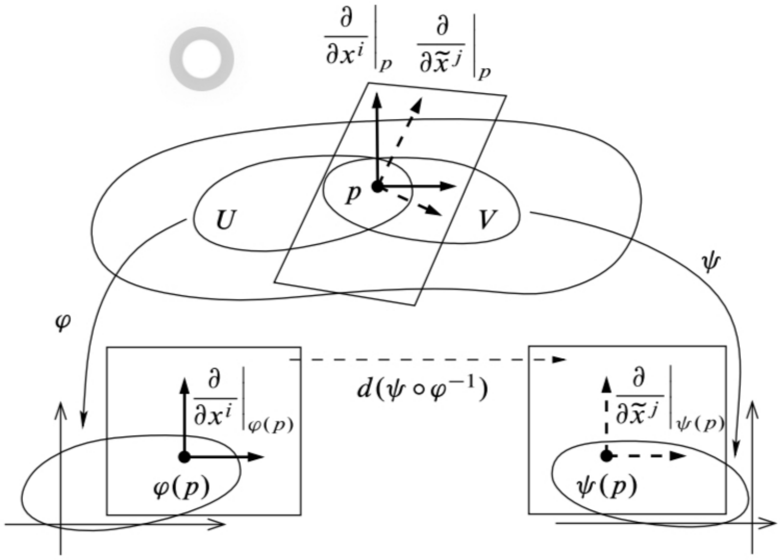

Figure 4.3: Change of coordinates

Using the definition of coordinate vectors, we obtain

$$
\begin{aligned}
\frac{\partial}{\partial x^i}\bigg|_p &= d\varphi^{-1}_{\varphi(p)}\left(\frac{\partial}{\partial x^i}\bigg|_{\varphi(p)}\right)\\
&= d\psi^{-1}_{\psi(p)} \circ d(\psi\circ\varphi^{-1})_{\varphi(p)}\left(\frac{\partial}{\partial x^i}\bigg|_{\varphi(p)}\right)\\
&= d\psi^{-1}_{\psi(p)}\left(\frac{\partial \tilde{x}^j}{\partial x^i}(\varphi(p))\ \frac{\partial}{\partial \tilde{x}^j}\bigg|_{\psi(p)}\right)\\
&= \frac{\partial \tilde{x}^j}{\partial x^i}(\varphi(p))\ \frac{\partial}{\partial \tilde{x}^j}\bigg|_p
\end{aligned}
$$

where we again write $p_y = \varphi(p)$. This formula resembles the standard chain rule in $\mathbb{R}^n$.
Now applying this to the components of a vector

$$v = v^i\ \frac{\partial}{\partial x^i}\bigg|_p = \tilde{v}^j\ \frac{\partial}{\partial \tilde{x}^j}\bigg|_p ,$$

we find that the components of v transform by the rule

$$\tilde{v}^j = \frac{\partial \tilde{x}^j}{\partial x^i}(p_y)\, v^i.$$

Example 4.6. (Polar to Cartesian Coordinates)
The transition map between polar coordinates and Cartesian coordinates is given by:

$$(x, y) = (r\cos\theta, r\sin\theta)$$

Let $p \in \mathbb{R}^2$ be the point whose polar coordinates are:

$$(r, \theta) = \left(2, \frac{\pi}{2}\right) \Rightarrow (x, y) = (0, 2)$$

Let the tangent vector in polar coordinates be:

$$v = 3\, \frac{\partial}{\partial r}\bigg|_p + \frac{\partial}{\partial \theta}\bigg|_p$$

Applying above equation to the coordinate vectors, we find:

$$\frac{\partial}{\partial r}\bigg|_p = \cos\left(\frac{\pi}{2}\right) \frac{\partial}{\partial x}\bigg|_p + \sin\left(\frac{\pi}{2}\right) \frac{\partial}{\partial y}\bigg|_p = \frac{\partial}{\partial y}\bigg|_p$$

$$\frac{\partial}{\partial \theta}\bigg|_p = -2\sin\left(\frac{\pi}{2}\right) \frac{\partial}{\partial x}\bigg|_p + 2\cos\left(\frac{\pi}{2}\right) \frac{\partial}{\partial y}\bigg|_p = -2\, \frac{\partial}{\partial x}\bigg|_p$$

Thus v has the following coordinate representation in standard coordinates:

$$v = 3\, \frac{\partial}{\partial y}\bigg|_p + 2\, \frac{\partial}{\partial x}\bigg|_p$$

.

4.5 The Tangent Bundle

Often it is useful to consider the set of all tangent vectors at all points of a manifold. Given a smooth manifold M (with or without boundary), we define the **tangent bundle** of M, denoted by TM, to be the disjoint union of the tangent spaces at all points of M:

$$TM = \bigsqcup_{p \in M} T_pM.$$

We usually write an element of this disjoint union as an ordered pair (p, v), with $p \in M$ and $v \in T_pM$ (instead of putting the point p in the second position, as elements of a disjoint union are more commonly written).
The tangent bundle comes equipped with a natural **projection map**

$$\pi : TM \to M,$$

which sends each vector in T_pM to the point p at which it is tangent: $\pi(p, v) = p$.
We will often commit the usual mild sin of identifying T_pM with its image under the canonical injection $v \mapsto (p, v)$, and will use any of the notations (p, v), v_p, or simply v for a tangent vector in T_pM, depending on how much emphasis we wish to give to the point p.

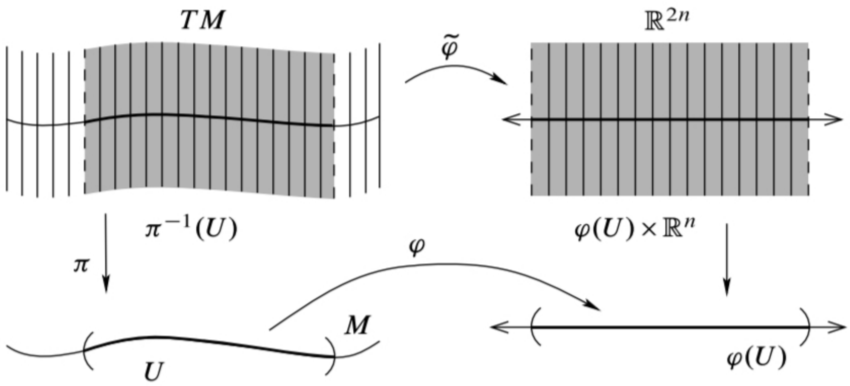

Figure 4.4: Coordinates for the tangent bundle

For example, in the special case $M = \mathbb{R}^n$, the tangent bundle of $\mathbb{R}^n$ can be canonically identified with the union of its geometric tangent spaces, which in turn is just the Cartesian product of $\mathbb{R}^n$ with itself:

$$T\mathbb{R}^n = \bigsqcup_{a \in \mathbb{R}^n} T_a\mathbb{R}^n \cong \bigsqcup_{a \in \mathbb{R}^n} \{a\} \times \mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n.$$

An element (a, v) of this Cartesian product can be thought of as representing either the geometric tangent vector v_a, or the derivation $D_v|_a$. However, that in general the tangent bundle of a smooth manifold cannot be identified in any natural way with a Cartesian product, because there is no canonical way to identify tangent spaces at different points with each other.

Chapter 5

RIEMANNIAN MANIFOLDS

To extend these geometric ideas to abstract smooth manifolds, we define a structure that amounts to a smoothly varying choice of inner product on each tangent space.
Let M be a smooth manifold. A **Riemannian metric** on M is a smooth covariant 2-tensor field $g \in T^2(M)$ whose value g_p at each $p \in M$ is an inner product on T_pM; thus g is a symmetric 2-tensor field that is positive definite in the sense that

$$g_p(v, v) \geq 0$$

for each $p \in M$ and each $v \in T_pM$, with equality if and only if $v = 0$.
A **Riemannian manifold** is a pair (M, g), where M is a smooth manifold and g is a specific choice of Riemannian metric on M. If M is understood to be endowed with a specific Riemannian metric, we sometimes say "M is a Riemannian manifold."
If M is a smooth manifold with boundary, a **Riemannian metric** on M is defined in exactly the same way as for manifolds without boundary: it is a smooth symmetric 2-tensor field g that is positive definite everywhere.
A **Riemannian manifold with boundary** is a pair (M, g), where M is a smooth manifold with boundary and g is a Riemannian metric on M.

Proposition 5.1. Every smooth manifold admits a Riemannian metric.

Proof. Let M be a smooth manifold. Our goal is to construct a Riemannian metric on M, i.e., a smooth choice of inner product on each tangent space T_pM that varies smoothly with p.
Since M is smooth, it admits an atlas of coordinate charts $\{(U_\alpha, \varphi_\alpha)\}$, where each $\varphi_\alpha : U_\alpha \to \mathbb{R}^n$ is a diffeomorphism onto its image.

On each chart U_α, define a Riemannian metric g_α by pulling back the standard Euclidean inner product from $\mathbb{R}^n$. That is,

$$g_\alpha = \varphi_\alpha^*(\delta),$$

where δ denotes the standard inner product on $\mathbb{R}^n$. Each g_α is then a smooth, positive-definite inner product on the tangent spaces over U_α.
Since $\{U_\alpha\}$ is an open cover of M, we can choose a smooth partition of unity $\{\psi_\alpha\}$ subordinate to this cover. That means each $\psi_\alpha : M \to [0,1]$ is a smooth function with support contained in U_α, and

$$\sum_\alpha \psi_\alpha(p) = 1 \quad \text{for all } p \in M,$$

with only finitely many nonzero terms in the sum at each point.
We then define a global Riemannian metric g on M by

$$g = \sum_\alpha \psi_\alpha g_\alpha.$$

At each point $p \in M$, this sum is finite and gives a convex combination of inner products, which remains a positive-definite symmetric bilinear form. Since each ψ_α and g_α is smooth, the resulting metric g is also smooth. Thus, g defines a Riemannian metric on M, and the proposition is proved. □

Definition 5.2. (Isometries and Local Isometries)
Suppose (M, g) and $(\widetilde{M}, \widetilde{g})$ are Riemannian manifolds with or without boundary. An **isometry** from (M, g) to $(\widetilde{M}, \widetilde{g})$ is a diffeomorphism

$$\varphi : M \to \widetilde{M}$$

such that

$$\varphi^*\widetilde{g} = g.$$

Unwinding the definitions shows that this is equivalent to requiring that φ preserves inner products between tangent vectors.
If (M, g) and $(\widetilde{M}, \widetilde{g})$ are Riemannian manifolds, a map

$$\varphi : M \to \widetilde{M}$$

is a **local isometry** if each point $p \in M$ has a neighborhood $U \subset M$ such that $\varphi|_U$ is an isometry onto an open subset of $\widetilde{M}$.

Proposition 5.3. *Prove that if* (M, g) *and* $(\widetilde{M}, \widetilde{g})$ *are Riemannian manifolds of the same dimension, a smooth map* $\varphi : M \to \widetilde{M}$ *is a local isometry if and only if*

$$\varphi^* \widetilde{g} = g.$$

Proof. Suppose first that φ is a local isometry. Then by definition, for each point $p \in M$, there exists a neighborhood $U \subseteq M$ containing p such that the restriction $\varphi|_U : U \to \varphi(U) \subseteq \widetilde{M}$ is a diffeomorphism and an isometry onto its image. In particular, for every $q \in U$ and for all tangent vectors $v, w \in T_q M$, we have

$$g_q(v, w) = \widetilde{g}_{\varphi(q)}(d\varphi_q(v), d\varphi_q(w)).$$

This equation says exactly that $\varphi^* \widetilde{g} = g$ on U. Since p was arbitrary and M can be covered by such neighborhoods U, it follows that $\varphi^* \widetilde{g} = g$ on all of M.

Conversely, suppose that $\varphi^* \widetilde{g} = g$. Then for every point $p \in M$ and for all $v, w \in T_p M$, we have:

$$g_p(v, w) = \widetilde{g}_{\varphi(p)}(d\varphi_p(v), d\varphi_p(w)).$$

This condition tells us that $d\varphi_p : T_p M \to T_{\varphi(p)} \widetilde{M}$ is an isometry of inner product spaces. Since M and $\widetilde{M}$ have the same dimension, $d\varphi_p$ is a linear isomorphism. Therefore, φ is a local diffeomorphism, and near each point $p \in M$, it is an isometry onto its image. Hence, φ is a local isometry.

$\square$

Definition 5.4. (Orthonormal Frames)
A local frame (E_i) for M on an open set U is said to be an **orthonormal frame** if the vectors $E_1|_p, \dots, E_n|_p$ are an orthonormal basis for $T_p M$ at each $p \in U$.

Proposition 5.5. (Existence of Orthonormal Frames) Let (M, g) be a Riemannian n-manifold with or without boundary. If (X_j) is any smooth local frame for TM over an open subset $U \subseteq M$, then there exists a smooth orthonormal frame (E_j) over U such that

$$\operatorname{span}(E_1|_p, \dots, E_k|_p) = \operatorname{span}(X_1|_p, \dots, X_k|_p)$$

for each $k = 1, \dots, n$ and each $p \in U$.
In particular, for every $p \in M$, there exists a smooth orthonormal frame (E_j) defined on some neighborhood of p.

Proof. Let $(X_1, \dots, X_n)$ be a smooth local frame for the tangent bundle TM over an open set $U \subseteq M$. At each point $p \in U$, the vectors $X_1|_p, \dots, X_n|_p$ form a basis for the tangent space T_pM. We aim to construct a new set of smooth vector fields $(E_1, \dots, E_n)$ over U such that:

1. Each E_j is smooth,
2. For each $p \in U$, the set $(E_1|_p, \dots, E_n|_p)$ is an orthonormal basis of T_pM with respect to g,
3. The span of $(E_1|_p, \dots, E_k|_p)$ equals the span of $(X_1|_p, \dots, X_k|_p)$ for all $k = 1, \dots, n$.

We apply the Gram–Schmidt process pointwise to the frame (X_j), using the inner product g_p at each point $p \in U$. Since the frame (X_j) and the metric g vary smoothly, and the Gram–Schmidt process involves only smooth operations (addition, scalar multiplication, inner products, square roots), the resulting orthonormal fields (E_j) are also smooth.
Therefore, we obtain a smooth orthonormal frame (E_j) on U that satisfies the required properties. In particular, for any point $p \in M$, such a frame exists in a neighborhood of p.

$\square$

5.1 Riemannian Submanifolds

Suppose $(\widetilde{M}, \widetilde{g})$ is a Riemannian manifold with or without boundary. Given a smooth immersion

$$F : M \to \widetilde{M},$$

the metric $g = F^*\widetilde{g}$ is called the *metric induced by* F.
On the other hand, if M is already endowed with a given Riemannian metric g, an immersion or embedding

$$F : M \to \widetilde{M}$$

satisfying $F^*\widetilde{g} = g$ is called an *isometric immersion* or *isometric embedding*, respectively.
Which terminology is used depends on whether the metric on M is considered to be given independently of the immersion or not.

The most important examples of induced metrics occur on submanifolds. Suppose $M \subseteq \widetilde{M}$ is an (immersed or embedded) submanifold, with or without boundary. The *induced metric* on M is the metric

$$g = \iota^* \widetilde{g}$$

induced by the inclusion map $\iota : M \hookrightarrow \widetilde{M}$. With this metric, M is called a *Riemannian submanifold* (or *Riemannian submanifold with boundary*) of $\widetilde{M}$.

Proposition 5.6. Let $(\widetilde{M}, \widetilde{g})$ be a Riemannian manifold with or without boundary, M a smooth manifold with or without boundary, and $F : M \to \widetilde{M}$ a smooth map. Then the pullback $g = F^* \widetilde{g}$ defines a Riemannian metric on M if and only if F is an immersion.

Proof. (⇒) Suppose $g = F^* \widetilde{g}$ is a Riemannian metric on M.
By definition, this means that for each $p \in M$, the bilinear form g_p is symmetric, bilinear, and positive definite on T_pM. Recall that the pullback metric is defined by

$$g_p(v, w) = \widetilde{g}_{F(p)}(dF_p(v), dF_p(w)), \quad \text{for all } v, w \in T_pM.$$

Suppose for contradiction that dF_p is not injective. Then there exists $v \in T_pM$, $v \neq 0$, such that $dF_p(v) = 0$. It follows that

$$g_p(v, v) = \widetilde{g}_{F(p)}(dF_p(v), dF_p(v)) = \widetilde{g}_{F(p)}(0, 0) = 0,$$

which contradicts the positive definiteness of g_p since $v \neq 0$. Thus, dF_p must be injective, and F is an immersion.
(⇐) Suppose F is an immersion.
Then for each $p \in M$, the differential dF_p is an injective linear map from T_pM to $T_{F(p)}\widetilde{M}$. Since $\widetilde{g}$ is a Riemannian metric on $\widetilde{M}$, the pullback metric defined by

$$g_p(v, w) = \widetilde{g}_{F(p)}(dF_p(v), dF_p(w))$$

is symmetric, bilinear, and positive definite on T_pM, because dF_p is injective and $\widetilde{g}$ is positive definite. The smoothness of g follows from the smoothness of F and $\widetilde{g}$.
Therefore, $g = F^* \widetilde{g}$ is a Riemannian metric on M.

□

Example 5.7. (Spheres) For each positive integer n, the unit n-sphere $S^n \subset \mathbb{R}^{n+1}$ is an embedded n-dimensional submanifold. The Riemannian metric induced on S^n by the Euclidean metric is denoted by $\widetilde{g}$ and is known as the *round metric* or *standard metric* on S^n.

Proposition 5.8. (Existence of Adapted Orthonormal Frames) Let $(\widetilde{M}, \widetilde{g})$ be a Riemannian manifold (without boundary), and let $M \subset \widetilde{M}$ be an embedded smooth submanifold with or without boundary. Given $p \in M$, there exists a neighborhood $\widetilde{U}$ of p in $\widetilde{M}$ and a smooth orthonormal frame for $\widetilde{M}$ on $\widetilde{U}$ that is adapted to M.

Proof. Let $p \in M$, and suppose $\dim \widetilde{M} = n$, $\dim M = k$, with $k < n$. Since M is an embedded submanifold of $\widetilde{M}$, we can choose a coordinate neighborhood $\widetilde{U}$ around p in $\widetilde{M}$ such that in these coordinates, the submanifold M is locally given by the vanishing of the last $n - k$ coordinates:

$$M \cap \widetilde{U} = \left\{ x \in \widetilde{U} \mid x^{k+1} = \cdots = x^n = 0 \right\}.$$

This means that the vector fields $\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^k}$ span the tangent space to M within $\widetilde{M}$, and the rest span the normal directions at points of M.
Now, consider any smooth local frame $\{X_1, \ldots, X_n\}$ on $\widetilde{U}$ such that the first k vector fields span TM along $M \cap \widetilde{U}$. Using the Gram–Schmidt process relative to the Riemannian metric $\widetilde{g}$, we can smoothly orthonormalize this frame to obtain an orthonormal frame $\{E_1, \ldots, E_n\}$ on $\widetilde{U}$. Since the Gram–Schmidt process preserves the span of the initial vectors, the first k vectors of the new frame $\{E_1, \ldots, E_k\}$ will span the tangent space of M at each point in $M \cap \widetilde{U}$.
Therefore, we have constructed a smooth orthonormal frame on a neighborhood of p in $\widetilde{M}$ such that the first k vectors span T_qM at each point $q \in M \cap \widetilde{U}$. This is exactly an orthonormal frame adapted to the submanifold M.

□

5.2 Pseudo-Riemannian Submanifold

Let $(\widetilde{M}, \widetilde{g})$ be a pseudo-Riemannian manifold. A submanifold $M \subset \widetilde{M}$ is called a *pseudo-Riemannian submanifold* if the metric g on M is the metric induced from $\widetilde{g}$ via the inclusion map $\iota : M \hookrightarrow \widetilde{M}$. That is, for all $p \in M$ and vectors $v, w \in T_pM$, we define:

$$g_p(v, w) = \widetilde{g}_p(v, w).$$

Then (M, g) is itself a pseudo-Riemannian manifold, and is called a *pseudo-Riemannian submanifold* of $(\widetilde{M}, \widetilde{g})$.

Proposition 5.9 (Orthonormal Frames for Pseudo-Riemannian Manifolds)**.** Let (M, g) be a pseudo-Riemannian manifold. For each $p \in M$, there exists a smooth orthonormal frame on a neighborhood of p in M.

Proof. Let $p \in M$ be arbitrary. Since M is a smooth manifold, there exists a smooth local frame $\{X_1, \ldots, X_n\}$ defined on a neighborhood $U \subseteq M$ of p. That is, for each point $q \in U$, the vectors $X_1|_q, \ldots, X_n|_q$ form a basis of the tangent space T_qM.
The metric g is a smooth, non-degenerate, symmetric $(0, 2)$-tensor field on M, so for each $q \in U$, the bilinear form g_q is non-degenerate and symmetric. By applying the Gram–Schmidt orthonormalization process adapted to pseudo-Riemannian geometry, we can construct at each point q an orthonormal basis $\{E_1|_q, \ldots, E_n|_q\}$ of T_qM with respect to g_q.
Furthermore, since the metric g and the original frame $\{X_j\}$ are smooth, the orthonormalization process can be performed smoothly to yield smooth vector fields $\{E_1, \ldots, E_n\}$ on U such that $g(E_i, E_j) = \varepsilon_i \delta_{ij}$, where $\varepsilon_i \in \{-1, 1\}$ depending on the signature of g.
Therefore, there exists a smooth orthonormal frame in a neighborhood of p. □

CONCLUSION

In this study, we have explored the foundational ideas behind the theory of topological and smooth manifolds, which play a central role in modern geometry. The project began with a detailed look at topological manifolds, focusing on the properties of topological spaces, such as the Hausdorff condition, second countability, and local Euclideanness. These conditions helped us define what makes a topological manifold, and we studied how charts and atlases help describe their local structure.
We then extended our study to smooth manifolds by introducing smooth atlases and the notion of compatibility between charts. This allowed us to define smooth functions on manifolds and understand how calculus can be generalized beyond Euclidean spaces. Important examples like spheres and projective spaces were used to illustrate these concepts and show how abstract definitions apply to real mathematical objects.
The project also discussed tangent spaces, differentials, and the construction of the tangent bundle, which gave us the tools to explore directions and variations on manifolds. Finally, we introduced Riemannian manifolds, where inner products are defined on tangent spaces in a smooth way, enabling us to talk about lengths, angles, and curvature.
Altogether, this project provides a strong foundation for further studies in areas such as differential geometry, topology, and mathematical physics. Understanding these basic ideas helps bridge the gap between geometry and analysis and prepares us to explore more advanced topics in modern mathematics.

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