ILL-POSED EQUATIONS

M.Sc. MATHEMATICS

2024-2026

ILL-POSED EQUATIONS

DISSERTATION

Submitted to the University of Kerala in partial fulfillment of the requirements for the award of Master of Science in Mathematics

Programme Code	:	620
Exam Code	:	62023402
Course Code	:	MM 545

CERTIFICATE

This is to certify that the dissertation entitled **"ILL-POSED EQUATIONS"** is a record of studies carried out by **AYISHA S**, M.Sc. Mathematics student of the year **2024–2026** at **T.K.M. College of Arts and Science, Kollam**, under my guidance and submitted to the University of Kerala in partial fulfillment of the Degree of Master of Science in Mathematics.

Kollam

July 2026

For

Mr. NISAMUDEEN ASHIQUE

Assistant Professor

Department of Mathematics

T.K.M. College of Arts and Science

Kollam

Dr. ASWATHY M R

H.O.D. and Assistant Professor

Department of Mathematics

T.K.M. College of Arts and Science

Kollam

Department of Mathematics
T.K.M.College of Arts and Science
Kollam-5

Contents

Introduction

Many mathematical models in science, engineering, and physics are expressed as operator equations of the form

$$Tx = y,$$

where T is an operator, x is the unknown solution, and y is the given data. In many practical applications, the objective is to determine the unknown quantity from indirect or noisy observations. Such problems are often ill-posed, meaning that they fail to satisfy one or more of the conditions of existence, uniqueness, or continuous dependence of the solution on the data.

An important class of ill-posed problems is formed by compact operator equations. A classical example is the backward heat conduction problem, in which the initial temperature distribution is reconstructed from later observations. Since small errors in the data may produce large errors in the solution, direct inversion is generally unstable.

To obtain meaningful approximate solutions, methods such as the Linear Least Residual Norm (LRN) solution and the generalized inverse are used. These techniques play an important role in the study of inverse problems and provide stable approximations when an exact inverse does not exist.

This project presents the basic theory of ill-posed operator equations, compact operator equations, the backward heat conduction problem, the Linear Least Residual Norm solution, and the generalized inverse.

Chapter 1

PRELIMINARIES

1.1 Basic concepts and definition

Definition 1.1.1. *A* ***linear space*** *(or* vector space*) over a field* $\mathbb{F}$ *(where* $\mathbb{F} = \mathbb{R}$ *or* $\mathbb{C}$*) is a non-empty set* X *together with two operations, namely vector addition and scalar multiplication, satisfying the following axioms for all* $x, y, z \in X$ *and* $\alpha, \beta \in \mathbb{F}$*:*

1. $x + y = y + x$.
2. $(x + y) + z = x + (y + z)$.
3. *There exists an element* $0 \in X$ *such that* $x + 0 = x$.
4. *For every* $x \in X$*, there exists an element* $-x \in X$ *such that* $x + (-x) = 0$.
5. $\alpha(x + y) = \alpha x + \alpha y$.
6. $(\alpha + \beta)x = \alpha x + \beta x$.
7. $\alpha(\beta x) = (\alpha\beta)x$.
8. $1x = x$*, where* 1 *is the multiplicative identity in* $\mathbb{F}$.

Example 1.1.1. *The set* $\mathbb{R}^n$ *with the usual vector addition and scalar multiplication is a linear space over* $\mathbb{R}$.

Definition 1.1.2. *Let X be a vector space over $\mathbb{R}$ or $\mathbb{C}$. A function*

$$\|\cdot\| : X \to [0, \infty)$$

is called a ***norm*** *on X if, for every $x, y \in X$ and every scalar α, the following properties hold:*

1. ***Positive definiteness:***

$$\|x\| \geq 0, \quad \textit{and } \|x\| = 0 \iff x = 0.$$

2. ***Homogeneity:***

$$\|\alpha x\| = |\alpha| \, \|x\|.$$

3. ***Triangle inequality:***

$$\|x + y\| \leq \|x\| + \|y\|.$$

A vector space equipped with a norm is called a normed vector space *(or simply a* normed space*).*

Definition 1.1.3. *A normed linear space is a vector space V together with a norm $\|\cdot\|$ such that for all $x, y \in V$ and any scalar α,*

1. ***Non-negativity:***

$$\|x\| \geq 0, \qquad \|x\| = 0 \iff x = 0.$$

2. ***Homogeneity:***

$$\|\alpha x\| = |\alpha| \, \|x\|.$$

3. ***Triangle Inequality:***

$$\|x + y\| \leq \|x\| + \|y\|.$$

If these three properties hold, then $(V, \|\cdot\|)$ is called a ***normed linear space****.*

Example 1.1.2. *Euclidean Space $\mathbb{R}^2$*

Let

$$V = \mathbb{R}^2 = \{(x, y) : x, y \in \mathbb{R}\}.$$

Define the norm by

$$\|(x, y)\| = \sqrt{x^2 + y^2}.$$

Definition 1.1.4. *Let X be a linear space over $\mathbb{F}$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. An* ***inner product*** *on X is a mapping*

$$\langle \cdot, \cdot \rangle : X \times X \to \mathbb{F}$$

such that, for all $x, y, z \in X$ and $\alpha \in \mathbb{F}$,

1. $\langle x, x \rangle \geq 0$, *and* $\langle x, x \rangle = 0$ *if and only if* $x = 0$.
2. $\langle x, y \rangle = \overline{\langle y, x \rangle}$.
3. $\langle \alpha x + y, z \rangle = \alpha \langle x, z \rangle + \langle y, z \rangle$.

Definition 1.1.5. *A linear space X equipped with an inner product $\langle \cdot, \cdot \rangle$ is called an* ***inner product space****.*

Definition 1.1.6. *A* ***Hilbert space*** *is an inner product space H that is complete with respect to the norm induced by the inner product, that is,*

$$\|x\| = \sqrt{\langle x, x \rangle}, \qquad x \in H.$$

Definition 1.1.7. *Let X and Y be linear spaces over the same field $\mathbb{F}$. A mapping*

$$T : X \to Y$$

is called a ***linear operator*** *if, for all $x, y \in X$ and $\alpha, \beta \in \mathbb{F}$,*

$$T(\alpha x + \beta y) = \alpha T(x) + \beta T(y).$$

Example 1.1.3. *Define $T : \mathbb{R}^2 \to \mathbb{R}^2$ by*

$$T(x, y) = (2x - y,\ x + 3y).$$

Then T is a linear operator.

Definition 1.1.8. *Let X and Y be normed linear spaces. A linear operator*

$$T : X \to Y$$

is called a ***bounded linear operator****. If there exists a constant $M \geq 0$ such that*

$$\|Tx\| \leq M\|x\|, \qquad \forall\, x \in X.$$

The set of all bounded linear operators from X into Y is denoted by

$$B(X,Y) = \{T : X \to Y : T \textit{ is a bounded linear operator}\}.$$

If $X = Y$, we write

$$B(X) = B(X,X).$$

Definition 1.1.9. *Let X and Y be normed linear spaces, and let $T : X \to Y$ be a bounded linear operator. The norm of T, called the* ***operator norm****, is defined by*

$$\|T\| = \sup\{\|Tx\| : \|x\| \le 1\}.$$

Equivalently,

$$\|T\| = \sup_{x \neq 0} \frac{\|Tx\|}{\|x\|}.$$

Definition 1.1.10. *Let $T : X \to Y$ be a linear operator. The* ***range*** *(or* image*) of T is defined by*

$$R(T) = \{Tx : x \in X\}.$$

That is, $R(T)$ is the set of all images of elements of X under the operator T.
Let $T : X \to Y$ be a linear operator. The ***null space*** *(or* kernel*) of T is defined by*

$$N(T) = \{x \in X : Tx = 0\}.$$

That is, $N(T)$ consists of all vectors in X that are mapped to the zero vector in Y.

Definition 1.1.11. *A linear operator $P : X \to X$ on a linear space X is called a* ***projection operator*** *or simply a projection if*

$$P(x) = x, \qquad \forall\, x \in R(P).$$

Thus, a linear operator$P : X \to X$is a projection operator if and only if $P^2 = P$

Definition 1.1.12. *Let H be a Hilbert space and let M be a closed subspace of H. A linear operator*

$$P : H \to H$$

is called the ***orthogonal projection*** *onto M if, for every $x \in H$, there exists a unique decomposition*

$$x = u + v,$$

where $u \in M$ *and* $v \in M^{\perp}$, *and*

$$Px = u.$$

Theorem 1.1.1. ***(Projection Theorem)*** *If* X_0 *is a complete subspace of an inner product space, then*

$$X = X_0 + X_0^{\perp} \quad \text{and} \quad \left(X_0^{\perp}\right)^{\perp} = X_0.$$

In particular, the above conclusion holds if X_0 *is a closed subspace of a Hilbert space* X. *Then, every* $x \in X$ *can be uniquely written as*

$$x = u + v \qquad \text{with } u \in X_0,\ v \in X_0^{\perp},$$

and in that case, $P : X \to X$ *defined by*

$$P(x) = u, \qquad x \in X,$$

is seen to be an orthogonal projection onto X_0, *i.e.,*

$$R(P) = X_0, \qquad N(P) = X_0^{\perp}.$$

Proof. Let $x \in X$. Since X_0 is a complete subspace of the inner product space X, there exists a unique element $u \in X_0$ such that

$$\|x - u\| = \inf\{\|x - z\| : z \in X_0\}.$$

Put

$$v = x - u.$$

We claim that $v \in X_0^{\perp}$.

Let $y \in X_0$ and $t \in \mathbb{R}$. Since u is the best approximation to x from X_0, we have

$$\|x - u\|^2 \leq \|x - (u + ty)\|^2.$$

Since $x - u = v$, this becomes

$$\|v\|^2 \leq \|v - ty\|^2.$$

Expanding,

$$\|v - ty\|^2 = \|v\|^2 - 2t\langle v, y\rangle + t^2\|y\|^2.$$

Hence,

$$-2t\langle v, y\rangle + t^2\|y\|^2 \geq 0$$

for every $t \in \mathbb{R}$. Therefore,

$$\langle v, y\rangle = 0$$

for every $y \in X_0$. Thus,

$$v \in X_0^{\perp}.$$

Hence every $x \in X$ can be written as,

$$x = u + v,$$

where $u \in X_0$ and $v \in X_0^{\perp}$.

To prove uniqueness, suppose

$$x = u_1 + v_1 = u_2 + v_2,$$

where $u_1, u_2 \in X_0$ and $v_1, v_2 \in X_0^{\perp}$. Then

$$u_1 - u_2 = v_2 - v_1.$$

The left-hand side belongs to X_0, while the right-hand side belongs to $X_0^{\perp}$. Hence

$$u_1 - u_2 \in X_0 \cap X_0^{\perp} = \{0\},$$

so $u_1 = u_2$ and consequently $v_1 = v_2$. Therefore,

$$X = X_0 \oplus X_0^{\perp}.$$

Also,

$$(X_0^{\perp})^{\perp} = X_0.$$

This completes the proof. □

Definition 1.1.13. *Let X and Y be normed linear spaces, X_0 be a subspace of X, and let*

$$T : X_0 \to Y$$

be a linear operator. The graph of T is defined by

$$G(T) = \{(x, Tx) : x \in X_0\} \subseteq X \times Y.$$

*The operator T is said to be a **closed operator** if its graph $G(T)$ is a closed subset of the product space $X \times Y$.*

Theorem 1.1.2. ***(Closed Graph Theorem)*** *Let X and Y be Banach spaces and X_0 be a subspace of X. Then a closed linear operator*

$$T : X_0 \to Y$$

is continuous if and only if X_0 is closed in X.

Definition 1.1.14. *Let X and Y be normed linear spaces and let*

$$T : X \to Y$$

*be a linear operator. The operator T is said to be a **compact operator** if for every bounded subset E of X, the set $T(E)$ has compact closure in Y.Equivalently, T is compact if for every bounded sequence $\{x_n\}$ in X, the sequence $\{Tx_n\}$ has a convergent subsequence in Y.*

Example 1.1.4. *Let $X = \ell^2$, and define the operator $T : \ell^2 \to \ell^2$ by*

$$T(x_1, x_2, x_3, \ldots) = \left(x_1, \frac{x_2}{2}, \frac{x_3}{3}, \ldots, \frac{x_n}{n}, \ldots\right).$$

Since $\frac{1}{n} \to 0$ as $n \to \infty$, the sequence of coefficients tends to zero. Hence, the image of every bounded sequence under T has a convergent subsequence. Therefore, T is a compact operator.

Chapter 2

ILL-POSED EQUATIONS

2.1 well posed equations

Let $T : X \to Y$ be a linear operator between normed linear spaces X and Y. The operator equation

$$Tx = y$$

is said to be well-posed if for every $y \in Y$, there exists a unique solution $x \in X$, and the solution depends continuously on y.
Equivalently, T is bijective and

$$T^{-1} : Y \to X$$

is continuous.

Example 2.1.1. *Consider the initial value problem*

$$\frac{dy}{dt} = 2y, \qquad y(0) = y_0.$$

Let

$$D : C[0,1] \longrightarrow C[0,1]$$

be defined by

$$D(f) = f', \qquad Dy = \frac{dy}{dt}.$$

(i) Existence Separate the variables:

$$\frac{dy}{dt} = 2y.$$

$$\frac{dy}{2y} = dt.$$

On integrating,

$$\int \frac{dy}{2y} = \int dt,$$

which gives

$$\frac{1}{2} \ln |y| = t + C.$$

Hence,

$$\ln |y| = 2t + C_1,$$

where $C_1 = 2C$. Therefore,

$$y = Ce^{2t}.$$

(ii) Uniqueness Using the initial condition,

$$y(0) = y_0,$$

we obtain

$$C = y_0.$$

Hence,

$$y(t) = y_0 e^{2t}.$$

(iii) Continuous Dependence on the Initial Data Let the initial value change from y_0 to $y_0 + \delta$. The corresponding solution is

$$\widetilde{y}(t) = (y_0 + \delta)e^{2t}.$$

Then

$$|\widetilde{y}(t) - y(t)| = |\delta|e^{2t}.$$

Hence,

$$\lim_{\delta \to 0} |\widetilde{y}(t) - y(t)| = 0.$$

Therefore, the initial value problem is well-posed.

2.2 Ill-Posed Equations

The operator equation

$$Tx = y$$

is said to be **ill-posed** if at least one of the conditions of well-posedness fails.

Example 2.2.1. *Backward heat conduction problem:*

$$\int_0^\pi k(s,t)x(t)\,dt = y(s), \qquad s \in [0,\pi],$$

where

$$k(s,t) = \frac{2}{\pi}\sum_{n=1}^{\infty} e^{-n^2}\sin(ns)\sin(nt).$$

Theorem 2.2.1. *Let $T : H \to H$ be a compact operator on a Hilbert space H. If $\{\varphi_n\}$ is an orthonormal basis and*

$$T\varphi_n = \delta_n\varphi_n,$$

then

$$\delta_n \to 0 \qquad \text{as } n \to \infty.$$

Theorem 2.2.2. *Let $T : H \to H$ be an operator. If there exists a constant $c > 0$ such that*

$$\|Tf\| \geq c\|f\| \qquad \text{for all } f \in H,$$

then T^{-1} is bounded.

2.3 A Backward Heat Conduction Problem

Let $u(s,t)$ represent the temperature at a point s on a thin wire of length ℓ at time t. Assuming that the wire is represented by the interval $[0,\ell]$ and its end-points are kept at zero temperature, it is known that $u(\cdot,\cdot)$ satisfies the partial differential equation

$$\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial s^2} \tag{2.1}$$

with boundary condition

$$u(0,t) = 0 = u(\ell,t). \tag{2.2}$$

Here, c represents the thermal conductivity of the material that the wire is made of. The problem that we discuss now is the following: Knowing the temperature at time $t = \tau$, say

$$g(s) := u(s,\tau), \qquad 0 < s \leq \ell,$$

determine the temperature distribution at an earlier time. determine the temperature at time $t = t_0 < \tau$, that is, find

$$f(s) := u(s,t_0), \qquad 0 < s < \ell.$$

Assume a solution of the heat equation of the form

$$u(s,t) = X(s)T(t)$$

Then

$$u_t(s,t) = X(s)T'(t)$$

$$u_s(s,t) = X'(s)T(t)$$

$$u_{ss}(s,t) = X''(s)T(t)$$

Substituting the separated solution

$$u(s,t) = X(s)T(t)$$

into Equation (2.1), we obtain

$$X(s)T'(t) = c^2X''(s)T(t).$$

Hence

$$\frac{T'(t)}{T(t)} = c^2 \frac{X''(s)}{X(s)},$$

or

$$\frac{T'(t)}{c^2 T(t)} = \frac{X''(s)}{X(s)} = -\lambda^2.$$

Therefore,

$$T'(t) + c^2\lambda^2 T(t) = 0 \tag{2.3}$$

Integrating,

$$\ln |T(t)| = -c^2\lambda^2 t + C,$$

and hence

$$T(t) = Ce^{-c^2\lambda^2 t}.$$

Similarly,

$$X''(s) + \lambda^2 X(s) = 0 \tag{2.4}$$

The characteristic equation is

$$m^2 + \lambda^2 = 0,$$

which gives,

$$m = \pm i\lambda.$$

Therefore,

$$X(s) = A\cos(\lambda s) + B\sin(\lambda s).$$

Hence,

$$u(s,t) = (A\cos(\lambda s) + B\sin(\lambda s))\, e^{-c^2\lambda^2 t}.$$

Applying the boundary conditions $u(0, t) = 0$ and $u(\ell, t) = 0$. we obtain,

$$0 = u(0, t) = Ae^{-c^2\lambda^2 t}.$$

Hence,

$$A = 0.$$

Therefore

$$u(s, t) = B\sin(\lambda s)e^{-c^2\lambda^2 t}.$$

Again, using the boundary condition $u(\ell, t) = 0$, we get

$$0 = u(\ell, t) = B\sin(\lambda\ell)e^{-c^2\lambda^2 t}.$$

For a non-trivial solution, we must have

$$\sin(\lambda\ell) = 0.$$

Thus,

$$\lambda\ell = n\pi, \qquad n \in \mathbb{N},$$

which implies

$$\lambda = \frac{n\pi}{\ell}.$$

Define

$$\lambda_n = \frac{cn\pi}{\ell}.$$

It can be seen easily that for each $n \in \mathbb{N}$,

$$u_n(s, t) := e^{-\lambda_n^2 t}\sin\left(\frac{n\pi s}{\ell}\right)$$

is a solution of the heat equation satisfying the boundary conditions. Let us assume that the general solution of the heat equation is of the form

$$u(s,t) = \sum_{n=1}^{\infty} a_n u_n(s,t).$$

Assuming that

$$f_0 = u(\cdot, 0) \in L^2([0,\ell]),$$

we have

$$f_0(s) = u(s,0) = \sum_{n=1}^{\infty} a_n u_n(s,0).$$

Since

$$u_n(s,t) = e^{-\lambda_n^2 t} \sin(\lambda_n s),$$

at $t = 0$,

$$u_n(s,0) = \sin(\lambda_n s).$$

Therefore,

$$f_0(s) = \sum_{n=1}^{\infty} a_n \sin(\lambda_n s).$$

Since $\sin(\lambda_n s)$ forms an orthogonal system on $[0,\ell]$, the coefficients a_n can be determined using the orthogonality of sine functions. To find the coefficients a_n, we consider the normalized eigenfunctions corresponding to $\sin(\lambda_n s)$.

$$\langle \sin(\lambda_n s), \sin(\lambda_m s) \rangle = 0, \qquad n \neq m,$$

and

$$\| \sin(\lambda_n s) \|^2 = \frac{\ell}{2}.$$

Hence,

$$\phi_n(s) = \sqrt{\frac{2}{\ell}} \sin(\lambda_n s), \qquad s \in [0,\ell],\ n \in \mathbb{N}.$$

Therefore,

$$f_0(s) = \sum_{n=1}^{\infty} a_n \sqrt{\frac{\ell}{2}}\, \phi_n(s).$$

and

$$u(s,t) = \sqrt{\frac{\ell}{2}} \sum_{n=1}^{\infty} a_n e^{-\lambda_n^2 t} \phi_n(s).$$

The set

$$\{\phi_n : n \in \mathbb{N}\}$$

is an orthonormal basis of $L^2([0,\ell])$. We have,

$$f_0(s) = \sum_{n=1}^{\infty} \langle f_0, \phi_n \rangle \phi_n(s).$$

Comparing this expansion with

$$f_0(s) = \sum_{n=1}^{\infty} a_n \sqrt{\frac{\ell}{2}}\, \phi_n(s),$$

we get

$$a_n \sqrt{\frac{\ell}{2}} = \langle f_0, \phi_n \rangle.$$

Hence,

$$a_n = \sqrt{\frac{2}{\ell}}\, \langle f_0, \phi_n \rangle.$$

Substituting the coefficient a_n into the series representation of the solution,

$$u(s,t) = \sum_{n=1}^{\infty} e^{-\lambda_n^2 t} \langle f_0, \phi_n \rangle \phi_n(s).$$

Since

$$|\phi_n(s)| \le \sqrt{\frac{2}{\ell}}, \qquad s \in [0,\ell],$$

by the Cauchy–Schwarz inequality,

$$\sum_{n=1}^{\infty} \left| e^{-\lambda_n^2 t} \langle f_0, \phi_n \rangle \phi_n(s) \right| \le \frac{2}{\ell} \|f_0\|_2 \left(\sum_{n=1}^{\infty} e^{-2\lambda_n^2 t} \right)^{1/2}.$$

for every $s \in [0,\ell]$ and every $t > 0$. Hence $u(s,t)$ is well-defined for every $s \in [0,\ell]$ and for every $t > 0$. Since

$$g(s) = u(s,\tau) = \sum_{n=1}^{\infty} e^{-\lambda_n^2 \tau} \langle f_0, \phi_n \rangle \phi_n(s), \tag{2.5}$$

we have,

$$\langle g, \phi_n \rangle = e^{-\lambda_n^2 \tau} \langle f_0, \phi_n \rangle.$$

Therefore,

$$\langle f_0, \phi_n \rangle = e^{\lambda_n^2 \tau} \langle g, \phi_n \rangle, \qquad n \in \mathbb{N}.$$

Hence the data g must satisfy the condition

$$\sum_{n=1}^{\infty} e^{2\lambda_n^2 \tau} \left| \langle g, \phi_n \rangle \right|^2 < \infty.$$

The above requirement can be interpreted as an additional smoothness condition that the data g must satisfy. In particular, we can conclude that the existence of a solution f is not guaranteed under the sole assumption that the data g belongs to $L^2([0,\ell])$. We have,

$$f(s) = u(s,t_0) = \sum_{n=1}^{\infty} e^{-\lambda_n^2 t_0} \langle f_0, \phi_n \rangle \phi_n(s), \qquad s \in [0,\ell].$$

So that

$$\langle f, \phi_n \rangle = e^{-\lambda_n^2 t_0} \langle f_0, \phi_n \rangle, \qquad n \in \mathbb{N}.$$

Consequently, from (2.5) we obtain

$$g = \sum_{n=1}^{\infty} e^{-\lambda_n^2 (\tau - t_0)} \langle f, \phi_n \rangle \phi_n.$$

Thus, the problem of finding f from the knowledge of g is equivalent to that of solving the operator equation

$$Kf = g.$$

where,

$$Kf = \sum_{n=1}^{\infty} e^{-\lambda_n^2 (\tau - t_0)} \langle f, \phi_n \rangle \phi_n, \qquad f \in L^2([0,\ell]).$$

By theorem 2.2.2 , Suppose there exists a constant $c > 0$ such that

$$\|Kf\| \geq c\|f\| \qquad \text{for all } f.$$

Then K^{-1} is bounded.

let

$$f = \varphi_n.$$

Since $\{\varphi_n\}$ is an orthonormal basis,

$$\|\varphi_n\| = 1.$$

Hence,

$$\|K\varphi_n\| \geq c\|\varphi_n\| = c.$$

Therefore,

$$\|K\varphi_n\| \geq c.$$

By theorem 2.2.1, But

$$K\varphi_n = \delta_n\varphi_n,$$

so,

$$\|K\varphi_n\|^2 = \delta_n^2.$$

Thus,

$$\|K\varphi_n\| = \delta_n.$$

Since

$$\delta_n \to 0,$$

we get

$$\|K\varphi_n\| \to 0.$$

Hence,

$$\delta_n \geq c,$$

$$\delta_n \to 0,$$

Therefore, K^{-1} is not bounded.

$$e^{-\lambda_n^2(\tau - t_0)} \to 0 \qquad \text{as } n \to \infty,$$

the above K is a compact operator from $L^2([0,\ell])$ to itself, and

$$\delta_n = e^{-\lambda_n^2(\tau - t_0)}, \qquad n \in \mathbb{N},$$

are the singular values of K. Thus K is not only not onto but does not have a continuous inverse from its range. Thus, the backward heat conduction problem considered is an ill-posed problem.

Chapter 3

LRN Solution and Generalized Inverse

3.1 LRN Solution

Let X and Y be linear spaces and $T : X \to Y$ be a linear operator. For $y \in Y$, the operator equation

$$Tx = y$$

has a solution if and only if $y \in R(T)$. If $y \notin R(T)$, then we look for an element $x_0 \in X$ such that Tx_0 is closest to y. For talking about closeness we require Y to be endowed with a metric. We assume that Y is a normed linear space. Suppose Y is a normed linear space and $y \in Y$. Then we may look for an $x_0 \in X$ which minimizes the functional

$$x \mapsto \|Tx - y\|, \qquad x \in X,$$

that is, to find $x_0 \in X$ such that

$$\|Tx_0 - y\| = \inf\{\|Tx - y\| : x \in X\}.$$

If such an x_0 exists, then we call it a *least residual norm solution (LRN solution)* of the operator equation.

$$Tx = y.$$

Clearly, if $y \in R(T)$, then every solution of the equation

$$Tx = y$$

is an LRN solution. However, if $y \notin R(T)$, then an LRN solution need not exist.

Example 3.1.1. *Let $X = C[a,b]$ with norm $\|\cdot\|_2$, $Y = L^2[a,b]$, and let $T : X \to Y$ be the identity map. If we take $y \in Y$ with $y \notin X$, then, using the denseness of X in Y, it follows that*

$$\inf\{\|Tx - y\| : x \in X\} = \inf\{\|x - y\|_2 : x \in C[a,b]\} = 0,$$

but there is no $x_0 \in X$ such that

$$\|Tx_0 - y\| = \|x_0 - y\|_2 = 0.$$

Remark 3.1.1 (Best Approximation Theorem)**.** *Suppose X is an inner product space and X_0 is a complete subspace of X. Then for every $x \in X$, there exists a unique element $x_0 \in X_0$ such that*

$$\|x - x_0\| = \inf\{\|x - u\| : u \in X_0\}.$$

In fact, the element x_0 is given by

$$x_0 = Px,$$

where P is an orthogonal projection from X onto X_0.

Proof. Let P be an orthogonal projection from X onto X_0. Then, for every $u \in X_0$, we have

$$x - u = (x - Px) + (Px - u),$$

where $x - Px \in X_0^{\perp}$ and $Px - u \in X_0$. Hence, by the Pythagorean theorem,

$$\|x - u\|^2 = \|x - Px\|^2 + \|Px - u\|^2 \geq \|x - Px\|^2.$$

Thus,

$$\|x - Px\| = \inf\{\|x - u\| : u \in X_0\}.$$

If $x_1 \in X_0$ also satisfies

$$\|x - x_1\| = \inf\{\|x - u\| : u \in X_0\},$$

then

$$\|x - x_1\|^2 = \|(x - x_0) + (x_0 - x_1)\|^2 = \|x - x_0\|^2 + \|x_0 - x_1\|^2.$$

Since

$$\|x - x_1\| = \|x - x_0\|$$

it follows that

$$\|x_0 - x_1\|^2 = 0,$$

and hence $x_1 = x_0$. □

Theorem 3.1.1. *Let X be a linear space, Y be a normed linear space, and $T : X \to Y$ be a linear operator. If either*

(i) $R(T)$ is finite dimensional, or

(ii) Y is a Hilbert space and $R(T)$ is a closed subspace of Y,

then an LRN solution of the operator equation

$$Tx = y$$

exists for every $y \in Y$.

Proof. (i) Let $y \in Y$.Since $R(T)$ is finite dimensional, it has the best approximation property. To prove this, let X_0 be a finite-dimensional subspace of a normed linear space X, and let $x \in X$.

Define

$$S = \{u \in X_0 : \|x - u\| \leq \|x - v\|, \ \forall v \in X_0\}.$$

Define the function

$$f : S \to \mathbb{R}, \qquad f(u) = \|x - u\|.$$

We first show that S is closed.

Let $\{u_n\} \in S$ and suppose $u_n \to u$.

Since the norm is continuous,

$$\|x - u_n\| \to \|x - u\|.$$

Also,

$$\|x - u_n\| \leq \|x - v\|, \qquad \forall\, v \in X_0.$$

Taking limits,

$$\|x - u\| \leq \|x - v\|, \qquad \forall\, v \in X_0.$$

Hence, $u \in S$. Therefore, S is closed.

Next, we show that S is bounded.

For any $u \in S$,

$$\|u\| = \|u - x + x\| \leq \|u - x\| + \|x\|.$$

Since

$$\|x - u\| \leq \|x - v\|, \qquad \forall\, v \in X_0,$$

we obtain

$$\|u\| \leq \|x - v\| + \|x\|.$$

Hence, S is bounded.

Since X_0 is finite dimensional, every closed and bounded subset of X_0 is compact.

Also, f is continuous on S. Hence, f attains its minimum on S.

Therefore, there exists $x_0 \in S$ such that

$$f(x_0) \leq f(u), \qquad \forall\, u \in S.$$

Hence,

$$\|x - x_0\| = \inf\{\|x - u\| : u \in X_0\}.$$

Thus, X_0 has the best approximation property.

Applying this to $R(T)$, there exists $y_0 \in R(T)$ such that

$$\|y - y_0\| = \inf\{\|y - v\| : v \in R(T)\}.$$

Since $y_0 \in R(T)$, there exists $x_0 \in X$ such that

$$Tx_0 = y_0.$$

Hence,

$$\|Tx_0 - y\| = \|y_0 - y\| = \inf\{\|y - v\| : v \in R(T)\} = \inf\{\|Tx - y\| : x \in X\}.$$

Therefore, x_0 is an LRN solution.

□

(ii) Suppose X is an inner product space and X_0 is a complete subspace of X. Then, for every $x \in X$, there exists a unique element $x_0 \in X_0$ such that

$$\|x - x_0\| = \inf\{\|x - u\| : u \in X_0\}.$$

In fact, the element x_0 is given by

$$x_0 = Px,$$

where P is the orthogonal projection of X onto X_0.

Since $R(T)$ is closed, there exists $y_0 \in R(T)$ such that

$$\|y - y_0\| = \inf\{\|y - v\| : v \in R(T)\}.$$

Since $y_0 \in R(T)$, there exists $x_0 \in X$ satisfying

$$Tx_0 = y_0.$$

Hence,

$$\|y - Tx_0\| = \|y - y_0\| = \inf\{\|y - v\| : v \in R(T)\}.$$

Moreover, every element of $R(T)$ is of the form Tx for some $x \in X$. Therefore,

$$\inf\{\|y - v\| : v \in R(T)\} = \inf\{\|y - Tx\| : x \in X\}.$$

Consequently,

$$\|y - Tx_0\| = \inf\{\|y - Tx\| : x \in X\},$$

which shows that x_0 is a least residual norm (LRN) solution.

Theorem 3.1.2. *Let X be a linear space, Y be a Hilbert space and $T : X \to Y$ be a linear operator.*
Let $P : Y \to Y$ be the orthogonal projection onto $\overline{R(T)}$. For $y \in Y$, the following are equivalent:

(i) The equation $Tx = y$ has a least residual norm (LRN) solution.

(ii) $y \in R(T) + R(T)^{\perp}$.

(iii) The equation $Tx = Py$ has a solution.

Proof. Step 1 *(ii)* $\iff$ *(iii)*.

$$Py \in R(T) \iff y \in R(T) \oplus R(T)^{\perp}.$$

Suppose that

$$Py \in R(T).$$

Since P is the orthogonal projection onto $R(T)$, every vector can be written as

$$y = Py + (y - Py),$$

where

$$Py \in R(T)$$

and

$$y - Py \in N(P) = R(T)^{\perp}.$$

Hence,

$$y \in R(T) \oplus R(T)^{\perp}.$$

Conversely, suppose that

$$y \in R(T) \oplus R(T)^{\perp}.$$

Then there exists

$$u \in R(T) \quad \text{and} \quad w \in R(T)^{\perp}$$

such that

$$y = u + w.$$

Since

$$u \in R(T) \quad \text{and} \quad w \in R(T)^{\perp},$$

this is the orthogonal decomposition of y. By the uniqueness of the orthogonal projection,

$$Py = u.$$

Hence,

$$Py \in R(T) \iff y \in R(T) \oplus R(T)^{\perp}.$$

Now assume that

$$Tx = Py$$

has a solution. Then there exist $x \in X$ such that

$$Tx = Py.$$

Since

$$R(T) = \{Tx : x \in X\},$$

it follows that Py is of the form Tx. Hence,

$$Py \in R(T).$$

Conversely, if

$$Py \in R(T),$$

then by the definition of the range of T, there exist $x \in X$ such that

$$Tx = Py.$$

(i) $\Leftrightarrow$ **(iii)** $Tx - y$ has an LRN solution. $\Leftrightarrow$ The eq^n $Tx = Py$ has solution . Assume $Tx = y$ has LRN solution

$$\|Tx_0 - y\| = \inf \|Tx - y\|$$

$$\begin{aligned}\inf_{x \in X} \|Tx - y\| &= \inf_{v \in R(T)} \|v - y\| \\ &= \inf_{v \in \overline{R(T)}} \|v - y\|\end{aligned}$$

$$\begin{gathered}\inf\{\|Tx - y\| : x \in X\} = \inf\{\|v - y\| : v \in \overline{R(T)}\} \\ = \inf\{\|v - y\| : v \in \overline{R(T)}\} = \|Py - y\|\end{gathered}$$

Thus $Tx = y$ has LRN solution x_0 iff $\|Tx_0 - y\| = \|Py - y\|$ Now

$$Tx_0 - y = (Tx_0 - Py) + (Py - y)$$

where $Tx_0 - Py \in R(T)$ and $Py - y \in R(T)^{\perp}$. Therefore these vectors are orthogonal. By Pythagoras theorem,

$$\|Tx_0 - y\|^2 = \|Tx_0 - Py\|^2 + \|Py - y\|^2$$

Since $\|Tx_0 - Py\| = \|Py - y\|$
we obtain,

$$\|Py - y\|^2 = \|Tx_0 - y\|^2 - \|Tx_0 - Py\|^2$$

hence $\|Tx_0 - Py\|^2 = 0$, which implies $Tx_0 = Py$, i.e., the equation $Tx = Py$ has a solution. Thus (i) $\Longrightarrow$ (iii). Conversely, assume (iii) holds. Then there exist, $x_0 \in X$ such that $Tx_0 = Py$.

$$\|Tx_0 - y\| = \|Py - y\|$$

$$\|Tx_0 - y\| = \|Py - y\|$$

But $Py - y = \inf \|Tx - y\| = \|Tx_0 - y\|$ Hence,

$$\|Tx_0 - y\| = \inf\{\|Tx - y\| : x \in X\}$$

x_0 is an LRN solution of the equation $Tx = y$.

Thus (iii) $\Rightarrow$ (i). This completes the proof. $\square$

Remark 3.1.2. *Let X, Y and T be as in Theorem 3.1.2. For $y \in Y$, we denote the set of all LRN solutions of $Tx = y$ by Sy, i.e.,*

$$S_y := \left\{ x \in X : \|Tx - y\| = \inf_{u \in X} \|Tu - y\| \right\}.$$

S_y can be empty . By Theorem 3.1.2,

$$S_y \neq \emptyset \iff y \in R(T) + R(T)^{\perp}.$$

Corollary 3.1.1. *Let X, Y and T be as in Theorem 3.1.2, $y \in Y$ be such that $S_y \neq \emptyset$. If $x_0 \in S_y$, then*

$$S_y = \{x_0 + u : u \in N(T)\}.$$

In particular, if T is injective and $y \in R(T) + R(T)^{\perp}$, then $Tx = y$ has a unique LRN solution.

Proof. If x_0 is an LRN solution, then for every $u \in N(T)$, $x_0 + u$ is also an LRN solution. Since $u \in N(T)$, we have $Tu = 0$, and therefore

$$T(x_0 + u) = Tx_0 + Tu = Tx_0.$$

So the residual does not change:

$$\|T(x_0 + u) - y\| = \|Tx_0 - y\|.$$

Since x_0 gives the minimum residual, $x_0 + u$ also gives the same residual.

Thus $x_0 + N(T) \subseteq S_y$.

Now we prove $S_y \subseteq x_0 + N(T)$. Take any $x \in S_y$. Since x and x_0 are both LRN solutions, by Theorem 3.1.2,

$$Tx = Py = Tx_0.$$

Since T is linear, $T(x - x_0) = Tx - Tx_0 = 0$.

so $x - x_0 \in N(T)$.

Let $u = x - x_0$. Then $u \in N(T)$ and $x = x_0 + u$.

Thus $S_y \subseteq \{x_0 + u : u \in N(T)\}$. Therefore

$$S_y = \{x_0 + u : u \in N(T)\}.$$

This means every LRN solution is of the form $x = x_0 + u$,where $u \in N(T)$. Suppose T is injective, then $N(T) = \{0\}$.

$$\begin{aligned} S_y &= x_0 + N(T) \\ &= x_0 + \{0\} \\ &= \{x_0\}. \end{aligned}$$

So S_y contains only one element, x_0. Therefore, there is only one LRN solution.

□

Remark 3.1.3. *If T is not injective, then the set S_y of all LRN solutions of $Tx = y$ generally contains more than one element. In such cases, a unique LRN solution is obtained by imposing an additional restriction, namely, by minimizing a suitable non-negative functional ϕ over S_y. The domain of ϕ may be a proper subset of X. The simplest choice is*

$$\phi(x) = \|x\|, \qquad x \in X,$$

which yields the least norm (LN) solution. More generally, one may consider

$$\phi(x) = \|Lx\|, \qquad x \in D(L),$$

where $L : D(L) \to Z$ is a linear operator, $D(L)$ is a subspace of X, and Z is another normed linear space. In many applications, X, Y, and Z are function spaces, and L is a differential operator. Although this general framework is important, we shall *mainly consider the case where X and Y are Hilbert spaces and seek the element of S_y with minimum norm. The same theory also applies to minimizing $\|Lx\|$ over $D(L) \cap S_y$.*

Theorem 3.1.3. *Let X and Y be Hilbert spaces, X_0 be a subspace of X, and $T : X_0 \to Y$ be a linear operator with $N(T)$ closed in X. Let $y \in R(T) + R(T)^\perp$. Then there exists a unique $\hat{x} \in S_y$ such that*

$$\|\hat{x}\| = \inf_{x \in S_y} \|x\|.$$

In fact,

$$\hat{x} \in N(T)^\perp \quad \text{and} \quad \hat{x} = Qx_0,$$

where $Q : X \to X$ is the orthogonal projection onto $N(T)^\perp$ and x_0 is any element in S_y.

Proof. Let $y \in R(T) + R(T)^\perp$. Then by Theorem 3.1.2, the set S_y of all LRN solutions is nonempty. Let $x_0 \in S_y$. By Corollary 3.1.1, we have

$$S_y = \{x_0 + u : \; u \in N(T)\}.$$

Let

$$\hat{x} = Qx_0,$$

where $Q : X \to X$ is the orthogonal projection onto $N(T)^\perp$.
Then we have

$$x_0 = \hat{x} + u, \qquad u \in N(T).$$

This also implies that

$$\hat{x} = x_0 - u \in X_0.$$

Hence, by Theorem 3.1.2,

$$T\hat{x} = Tx_0 = Py,$$

where $P : Y \to Y$ is the orthogonal projection onto $\overline{R(T)}$.
Thus,

$$\hat{x} \in S_y.$$

Again, by Corollary 3.1.1, for any $x \in S_y$, we have,

$$x - \hat{x} \in N(T).$$

Since

$$Tx = Py \quad \text{and} \quad T\hat{x} = Py,$$

it follows that

$$\begin{aligned} T(x - \hat{x}) &= Tx - T\hat{x} \\ &= Py - Py \\ &= 0. \end{aligned}$$

Hence,

$$x - \hat{x} \in N(T).$$

We know that

$$\hat{x} \in N(T)^{\perp}.$$

Thus,

$$\hat{x} \perp (x - \hat{x}).$$

Also,

$$x = \hat{x} + (x - \hat{x}).$$

Hence, by the Pythagoras theorem,

$$\|x\|^2 = \|\hat{x}\|^2 + \|x - \hat{x}\|^2.$$

Thus,

$$\|\hat{x}\| \leq \|x\|, \qquad \forall\, x \in S_y.$$

Suppose there exists another element $\tilde{x} \in S_y$ such that

$$\|\tilde{x}\| \leq \|x\|, \qquad \forall\, x \in S_y.$$

Then it follows that,

$$\|\tilde{x}\| \leq \|\hat{x}\| \leq \|\tilde{x}\|,$$

so that

$$\|\tilde{x}\| = \|\hat{x}\|.$$

Using again Corollary 3.1.1 and the Pythagoras theorem,

we have,

$$\tilde{x} - \hat{x} \in N(T)$$

and

$$\|\tilde{x}\|^2 = \|\hat{x}\|^2 + \|\tilde{x} - \hat{x}\|^2.$$

Consequently,

$$\tilde{x} = \hat{x}.$$

□

Corollary 3.1.2. *Let X, X_0, Y and T be as in Theorem 3.1.3, and let $Q : X \to X$ be the orthogonal projection onto $N(T)^\perp$.*
Let $y \in R(T) + R(T)^\perp$. Then

$$\left\{x \in S_y : \|x\| = \inf_{u \in S_y} \|u\|\right\}, \qquad N(T)^\perp \cap S_y, \quad \text{and} \quad \{Qx : x \in S_y\}$$

are singleton sets.

Proof. By Theorem 3.1.3, there exists a unique $\hat{x} \in S_y$ such that

$$\{\hat{x}\} = \left\{x \in S_y : \|x\| = \inf_{u \in S_y} \|u\|\right\} = \{Qx : x \in S_y\}.$$

Thus,

$$\hat{x} \in N(T)^\perp \cap S_y.$$

Hence, it suffices to show that $N(T)^\perp \cap S_y$ does not contain more than one element. Suppose $x_1, x_2 \in N(T)^\perp \cap S_y$. Since $x_1, x_2 \in S_y$, by Theorem 3.1.2,

$$Tx_1 = Tx_2,$$

so that

$$x_1 - x_2 \in N(T).$$

Also,

$$x_1 - x_2 \in N(T)^\perp.$$

Therefore,

$$x_1 - x_2 \in N(T) \cap N(T)^\perp = \{0\}.$$

Hence,

$$x_1 = x_2.$$

Thus,

$$N(T)^\perp \cap S_y = \{\hat{x}\},$$

and the proof is complete. $\square$

3.2 Generalized inverse

Let X, Y be Hilbert spaces, X_0 be a subspace of X, and $T : X_0 \to Y$ be a linear operator with $N(T)$ closed in X. By Theorem 3.1.3, for every

$$y \in R(T) + R(T)^\perp,$$

the set

$$S_y := \left\{ x \in X_0 : \|Tx - y\| = \inf_{u \in X_0} \|Tu - y\| \right\}$$

of LRN solutions of $Tx = y$ is non-empty and there exists a unique $\hat{x} \in S_y$ such that

$$\|\hat{x}\| = \inf_{x \in S_y} \|x\|.$$

In fact, by Theorems 3.1.2 and 3.1.3, $\hat{x}$ is the unique element in $N(T)^\perp$ such that

$$T\hat{x} = Py,$$

where $P : Y \to Y$ is the orthogonal projection onto $\overline{R(T)}$. Let $T^\dagger$ be the map which associates each $y \in R(T) + R(T)^\perp$ to the unique LRN-solution of minimal norm. Then it can be seen that

$$T^\dagger : R(T) + R(T)^\perp \longrightarrow X$$

is a linear operator with dense domain

$$D(T^\dagger) := R(T) + R(T)^\perp.$$

The operator $T^\dagger$ is called the *generalized inverse* or *Moore–Penrose inverse* of T, and $\hat{x} := T^\dagger y$ is called the *generalized solution* of $Tx = y$. In fact,

$$T^\dagger y = \left(T\big|_{N(T)^\perp \cap X_0}\right)^{-1} Py, \qquad y \in D(T^\dagger).$$

Remark 3.2.1. *Let Y be a Hilbert space and Y_0 be a subspace of Y. Then Y_0 is closed if and only if*

$$Y_0 + Y_0^\perp = Y.$$

Remark 3.2.2. *Let X and Y be Hilbert spaces, X_0 be a subspace of X, and let*

$$T : X_0 \to Y$$

be a linear operator with $N(T)$ closed. Then:

(i) $D(T^\dagger)$ is dense in Y,

(ii) $T^\dagger$ is a linear operator,and

(iii) For every $y \in D(T^\dagger)$

$$T^\dagger y = \left(T\big|_{N(T)^\perp \cap X_0}\right)^{-1} Py,$$

where $P : Y \to Y$ is the orthogonal projection onto $\overline{R(T)}$.

Theorem 3.2.1. *Let X and Y be Hilbert spaces, X_0 be a subspace of X, and*

$$T : X_0 \to Y$$

be a closed linear operator. Then

(i) $T^\dagger$ is a closed linear operator,and

(ii) $T^\dagger$ is continuous if and only if $R(T)$ is a closed subspace of Y.

Proof. (i) To prove that $T^\dagger$ is closed, let $\{y_n\} \subset D(T^\dagger)$ be such that

$$y_n \to y \quad \text{and} \quad T^\dagger y_n \to x \text{ in } X.$$

By the definition of the generalized inverse,

$$T(T^\dagger y_n) = Py_n,$$

where P is the orthogonal projection onto $R(T)$. Since P is continuous and $y_n \to y$, we have,

$$Py_n \to Py.$$

Hence,

$$T(T^\dagger y_n) \to Py.$$

Since $T^\dagger y_n \to x$ and T is closed, it follows that

$$x \in X_0 \quad \text{and} \quad Tx = Py.$$

Therefore, by Theorem 3.1.2,

$$x \in S_y,$$

that is, x is a least residual norm solution corresponding to y. Since

$$Tx = Py,$$

we obtain

$$y \in R(T) \oplus R(T)^\perp = D(T^\dagger).$$

Moreover,

$$T^\dagger y = \left(T\big|_{N(T)^\perp \cap X_0}\right)^{-1} Py.$$

Since

$$T^\dagger y_n \in N(T)^\perp \cap X_0$$

for every n, and $N(T)^\perp$ is closed,

$$x = \lim_{n\to\infty} T^\dagger y_n \in N(T)^\perp.$$

Thus $x \in S_y \cap N(T)^\perp$. By Theorem 3.1.3, there is exactly one element of S_y lying in $N(T)^\perp$. Hence,

$$T^\dagger y = x.$$

Therefore,

$$(y, T^\dagger y) = (y, x)$$

belongs to the graph of $T^\dagger$, proving that the graph of $T^\dagger$ is closed. Hence $T^\dagger$ is a closed linear operator.

(ii) Since $T^\dagger$ is a closed linear operator, the Closed Graph Theorem implies that $T^\dagger$ is continuous if and only if its domain $D(T^\dagger)$ is closed. But,

$$D(T^\dagger) = R(T) \oplus R(T)^\perp.$$

Since $R(T)^\perp$ is always a closed subspace of Y, it follows that $D(T^\dagger)$ is closed if and only if $R(T)$ is closed. Hence,

$$T^\dagger \text{ is continuous} \iff R(T) \text{ is a closed subspace of } Y.$$

This completes the proof.

□

From Theorem 3.2.1, if $R(T)$ is not closed in Y, then the generalized inverse $T^\dagger$ is not continuous. Consequently, the problem of finding the generalized solution

$$\hat{x} = T^\dagger y$$

of the operator equation $Tx = y$ is ill-posed, since small perturbations in the data may produce large changes in the solution. This highlights the limitations of the generalized inverse for ill-posed problems and motivates the development of regularization methods, which provide stable approximations to the generalized solution.

3.3 Applications of Ill-Posed Problems

(i) **Medical Imaging:** Ill-posed problems arise in image reconstruction techniques such as Computed Tomography (CT), Magnetic Resonance Imaging (MRI), and Positron Emission Tomography (PET).

(ii) **Image Restoration:** They are used to recover blurred, noisy, or damaged images.

(iii) **Geophysics:** Ill-posed problems are encountered in determining the Earth's internal structure from seismic data.

(iv) **Signal Processing:** Ill-posed problems arise in reconstructing signals from noisy or incomplete observations.

(v) **Remote Sensing:** They are used to recover information about the Earth's surface from satellite and aerial data.

Conclusion

This project presented a study of ill-posed operator equations and the methods used to obtain meaningful solutions. The concepts of well-posed and ill-posed equations were discussed, and the backward heat conduction problem was examined as a classical example of an ill-posed problem, illustrating the instability caused by small perturbations in the data.

The Least Relative Norm (LRN) solution and the theory of generalized inverses were studied as effective tools for solving such problems. The generalized inverse extends the concept of the ordinary inverse and provides a suitable framework for obtaining meaningful solutions when a unique solution does not exist.

Overall, this study provides a basic understanding of ill-posed equations, the backward heat conduction problem, and the importance of generalized inverses in solving inverse problems arising in mathematics and its applications.

Bibliography

[1] B. V. Limaye, *Functional Analysis*, 2nd Edition, New Age International (P) Ltd., New Delhi, 1996.

[2] M. Thamban Nair, *Linear Operator Equations: Approximation and Regularization*, World Scientific Publishing Co. Pte. Ltd., Singapore, 2009.

[3] Haim Brezis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Springer, Universitext Series, 2011.

[4] Farheen M. Shaikh, *Convergence Analysis of Some Variants of Levenberg–Marquardt Iterative Method for Nonlinear Ill-Posed Operator Equations*, Ph.D. Thesis, Visvesvaraya National Institute of Technology, Nagpur, 2023.