AN INTRODUCTION TO MATROID THEORY

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AN INTRODUCTION TO MATROID THEORY

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CERTIFICATE

This is to certify that the dissertation entitled **"AN INTRODUCTION TO MATROID THEORY"** is a record of studies carried out by **ANJALI S AJAYAN**, M. Sc Mathematics student of the year **2024 - 2026** at **T. K. M. College of Arts and Science**, under my guidance and submitted to the University of Kerala in partial fulfilment of the Degree of Masters of Science in Mathematics.

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INTRODUCTION

Matroid theory is an important branch of modern mathematics that studies the concept of independence in a general and unified way. It combines ideas from linear algebra, graph theory, and combinatorics into a single mathematical framework. In linear algebra, independence refers to linearly independent vectors, while in graph theory it refers to sets of edges that do not contain cycles. The theory provides a simple and elegant way to understand many mathematical problems involving independence, dependence, rank, bases, and circuits.

This project introduces the basic concepts of matroid theory and explains how matroids generalize familiar ideas from vector spaces and graphs. It discusses the fundamental definitions, important properties, and different types of matroids, with special emphasis on graphic matroids and their relationship with graph theory. Because of its ability to unify different areas of mathematics, matroid theory has become an active field of research with applications in optimization, computer science, network theory, coding theory, and operations research.

1.1. HISTORY OF MATROID THEORY

The development of matroid theory began with the search for a common mathematical framework to describe the notion of independence. Before the twentieth century, the concept of independence appeared in different branches of mathematics in different forms. In linear algebra, independence referred to linearly independent vectors, while in graph theory it referred to edge sets that contain no cycles. Although these ideas arose independently, mathematicians recognized that they possessed many similar properties. This observation motivated the search for a single abstract theory capable of describing these different notions of independence.

The formal beginning of matroid theory dates to 1935, when the American mathematician Hassler Whitney introduced the concept of a matroid in his pioneering paper On the Abstract Properties of Linear Dependence, published in the American Journal of Mathematics. Whitney observed that many fundamental properties of linearly independent sets of vectors do not depend on vectors themselves but only on certain abstract axioms satisfied by those sets. He developed a new mathematical structure, called a matroid, to capture these common properties. Whitney also showed that graphs naturally give rise to matroids, now known as graphic matroids, establishing a deep connection between graph theory and linear algebra.

During the 1940s and 1950s, matroid theory was developed further by several mathematicians, especially W. T. Tutte investigated the structure of matroids, introduced important concepts such as connectivity and duality, and established deep connections between matroids and graph theory. Later, Jack Edmonds showed that matroids provide the mathematical foundation for greedy algorithms, making the subject highly relevant in combinatorial optimization.

1.2. OBJECTIVES OF THE PROJECT

- To study the basic concepts of matroid theory.
- To understand the concept of independence in matroids.
- To study independent sets, bases, circuits, rank functions, closure operators, and flats.
- To examine graphic matroids and their relationship with graphs.
- To study dual matroids and their basic properties.
- To explore applications of matroid theory in graph theory.

1.3. ORGANIZATION OF THE PROJECT

The introductory part of the project includes the history, objectives, and overall organization of the project.

Chapter 1 provides the preliminary concepts required for the study of matroids. It introduces basic graph theory, independent sets, trees and spanning trees, cycles and so on.

Chapter 2 discusses the fundamental concepts of matroid theory. It presents the definition of a matroid and studies independent sets, bases, circuits, rank functions, closure operators, and flats, together with suitable examples.

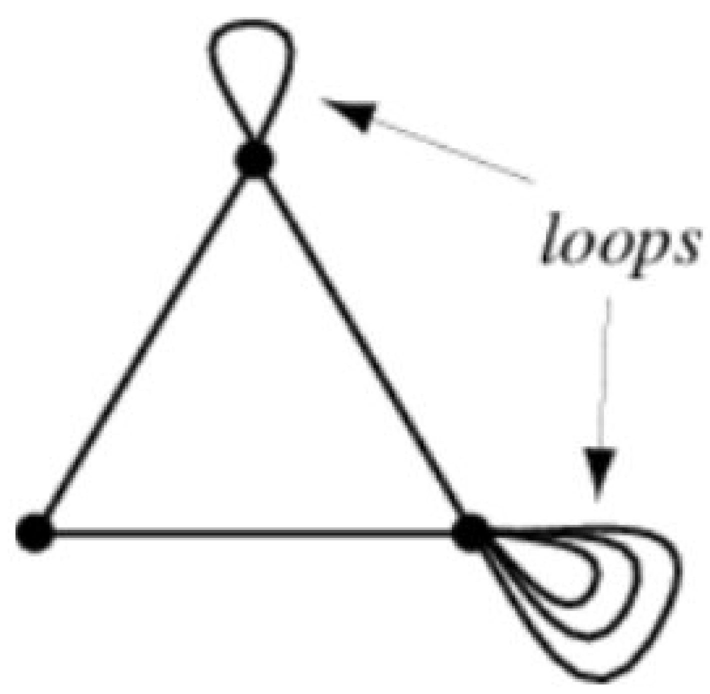
Chapter 3 focuses on graphic and dual matroids. It explains graphic matroids, cycle matroids, spanning trees as bases, circuits as cycles, the rank of a graphic matroid, and dual (cographic) matroids. It also discusses their properties, examples, and applications in graph theory, including \greedy algorithms, coding theory, network design. network optimization, and connectivity problems.

Conclusion summarizes the main concepts presented in the project. It also briefly discusses the future scope of matroid theory and its further applications.

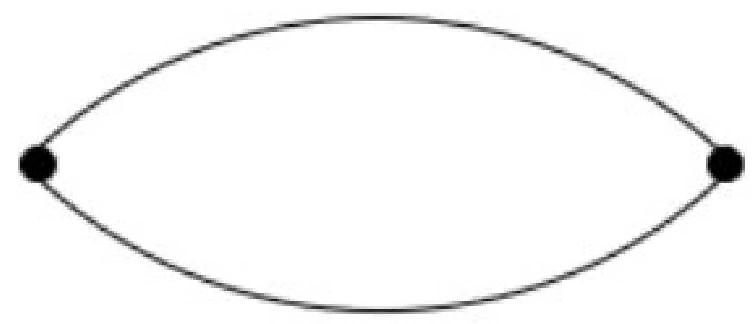
CHAPTER I

PRELIMINARIES

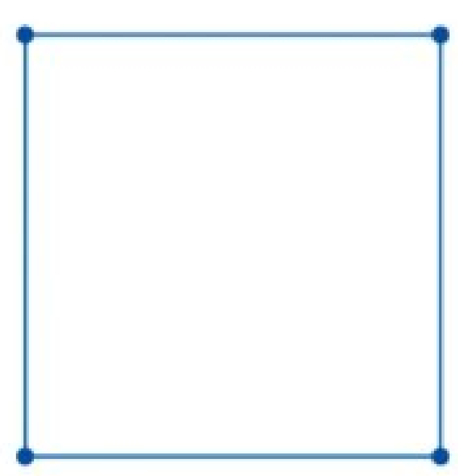
- GRAPH: A graph is a pair G = (V,E), where V is the set of vertices and E is the set of edges, formed by pair of vertices.
- If an edge connects to a vertex we say the edge is incident to the vertex and say the vertex is an endpoint of the edge.
- ADJACENT VERTICES: Two vertices that are joined by an edge are called adjacent vertices.
- LOOP: A loop is an edge whose endpoints are equal. Multiple edges are edges having the same pair of endpoints.



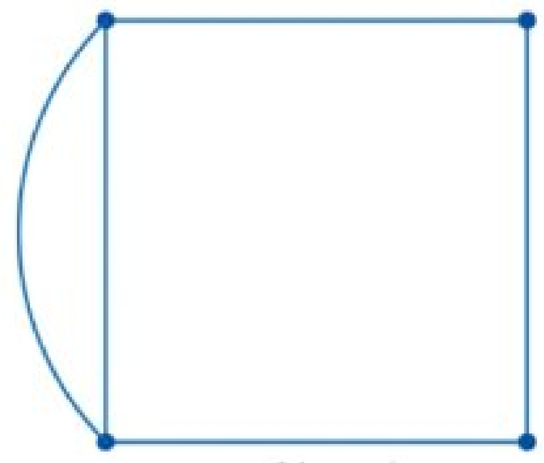
- MULTIPLE OR PARALLEL EDGES: If two or more edges have the same endpoints then they are called multiple or parallel edges.

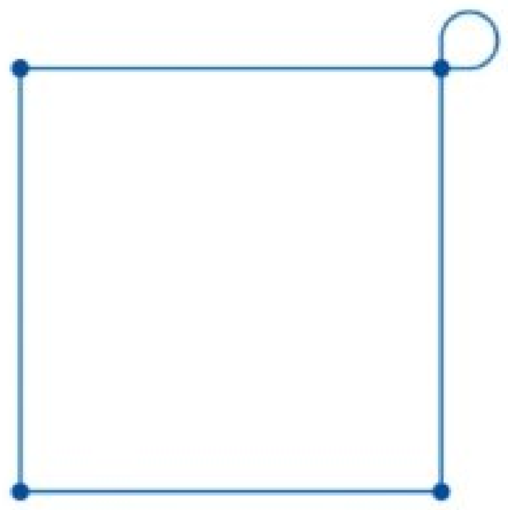


- PENDANT VERTEX: A pendant vertex is a vertex that is connected to exactly one other vertex by a single edge.
- SIMPLE GRAPH: A simple graph is a graph having no loops or multiple edges.

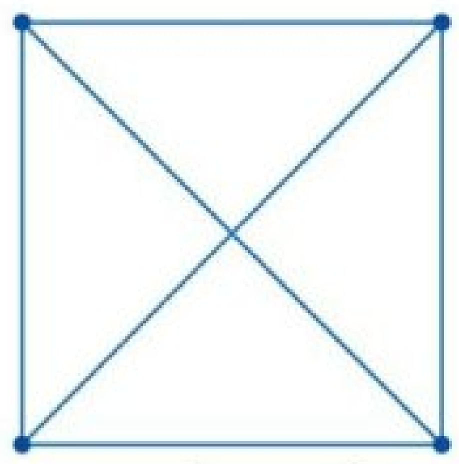


❖ A graph with more than one edge between a pair of vertices is called a multigraph while a graph with loop edges is called a pseudograph.

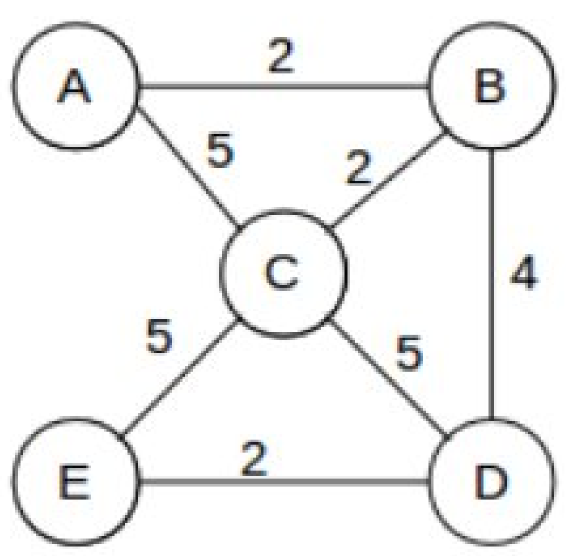




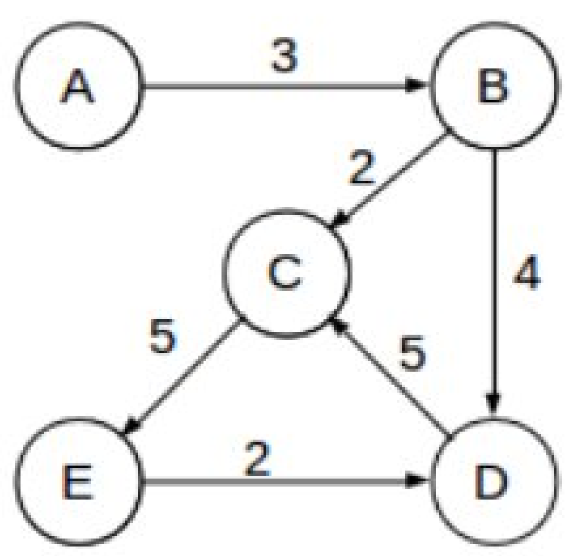
❖ COMPLETE GRAPH: A complete graph is a simple graph in which every pair of distinct vertices is joined by exactly one edge. The complete graph on n vertices is denoted by K_n.



❖ DIRECTED GRAPH: A directed graph is a graph in which the edges may only be traversed in one direction. Edges in a simple directed graph may be specified by an ordered pair (v_i, v_j) of the two vertices that the edge connects. We say that v_i is adjacent to v_j and v_j is adjacent from v_i.

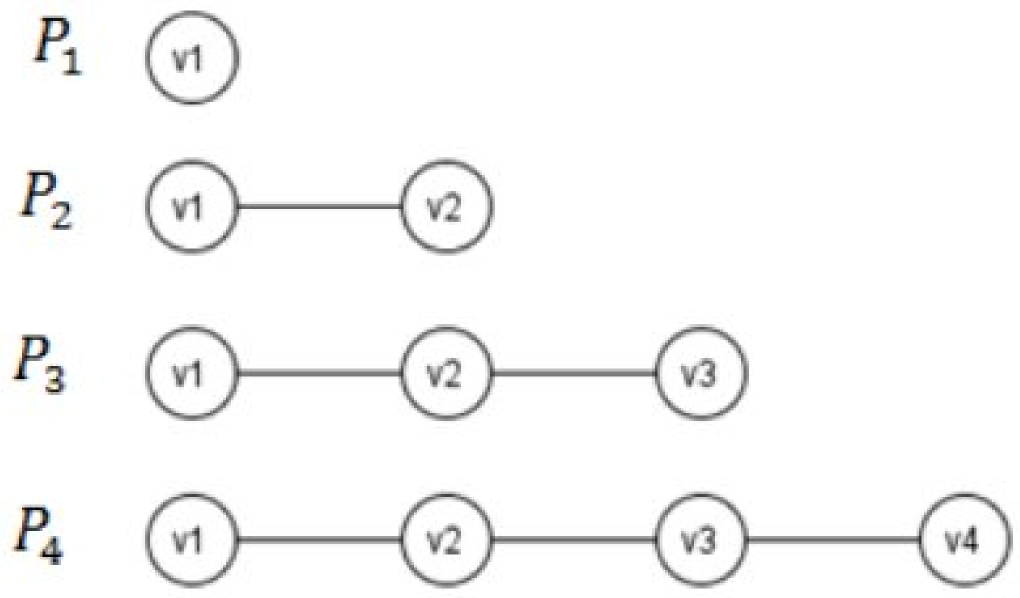


Undirected

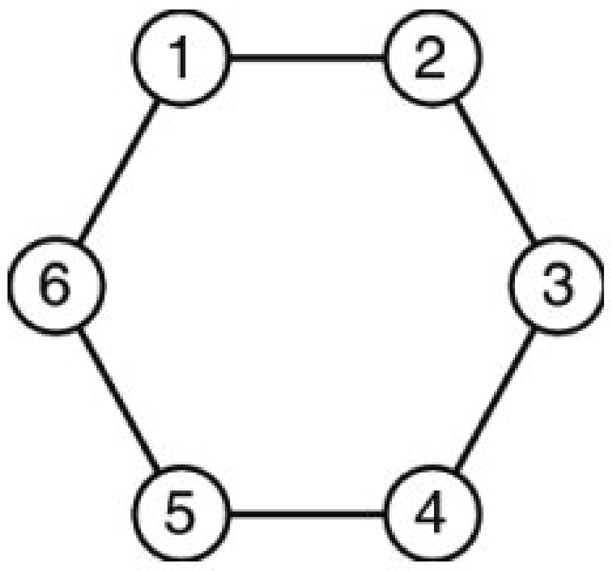


Directed

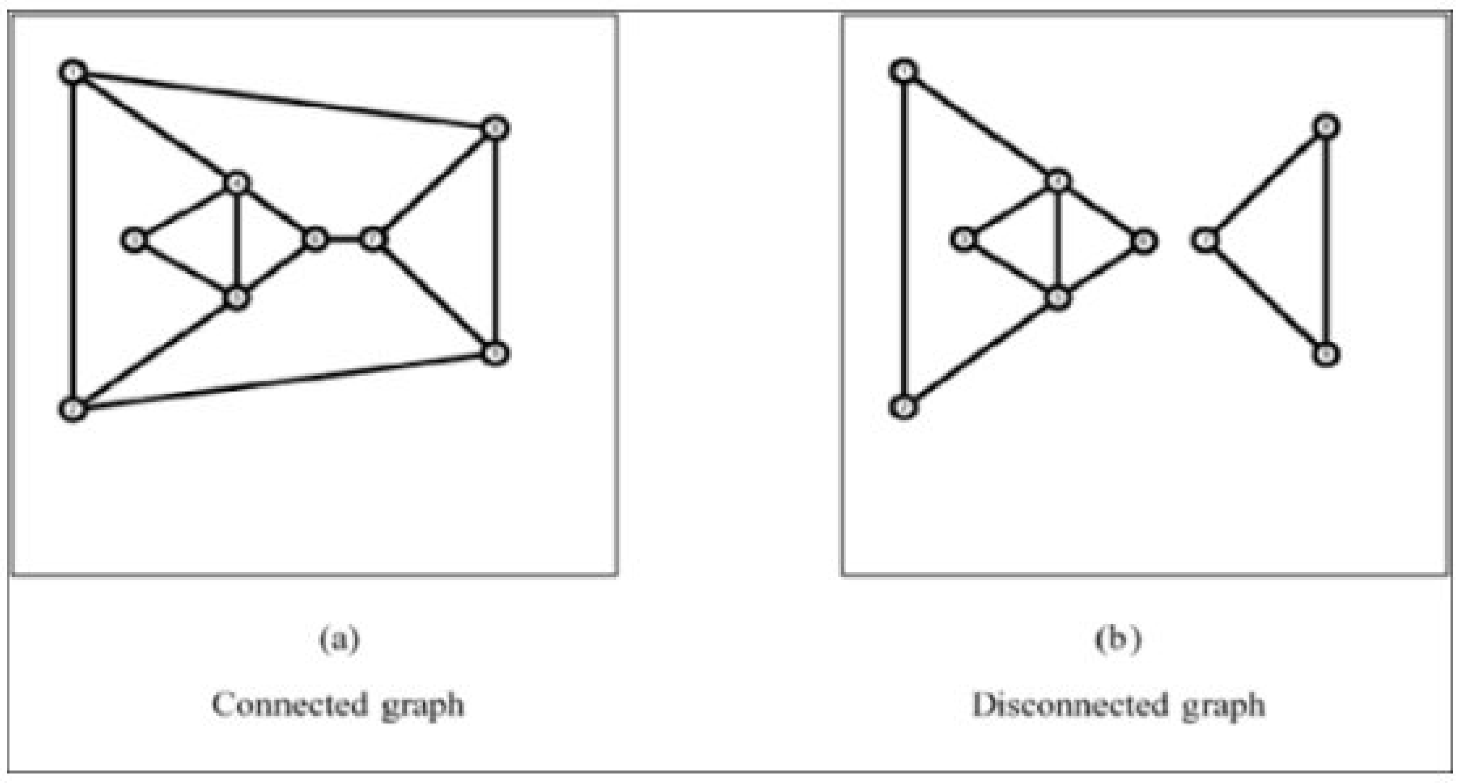
❖ PATH: A path is a simple graph whose vertices can be ordered so that two vertices are adjacent if and only if they are consecutive in the list.



❖ CYCLE: A cycle in a graph is a closed path in which no vertex is repeated except that the first and last vertices are the same.

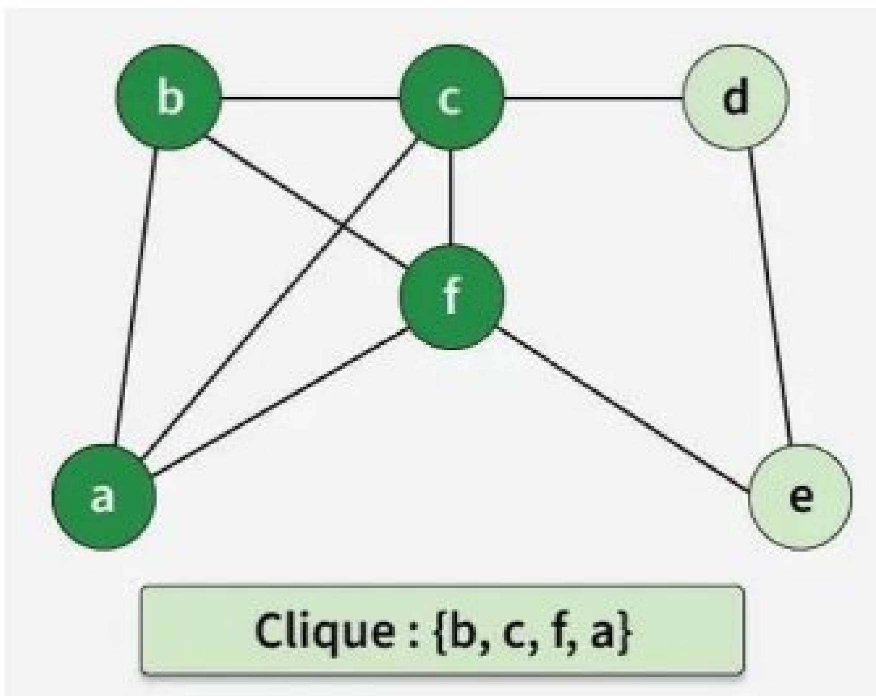


❖ CONNECTED GRAPH: A connected graph is a graph in which there exists a path between every pair of distinct vertices. Otherwise, the graph is called a disconnected graph.



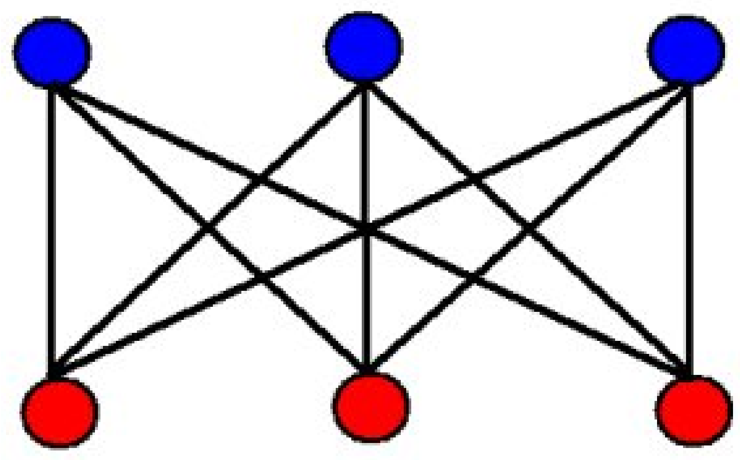
❖ INDEPENDENT SET: An independent set in a graph is a set of vertices no two of which are adjacent.

❖ CLIQUE: A clique in a graph is a set of pairwise adjacent vertices.



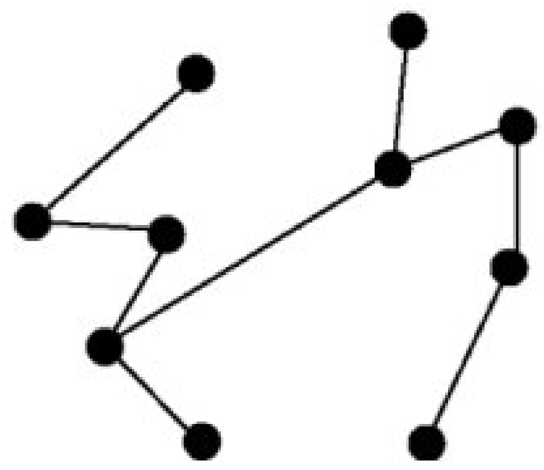
❖ MAXIMAL PATH: A maximal path in a graph is a path that cannot be extended by adding another vertex at either end while remaining a path.

❖ BIPARTITE GRAPH: A graph is said to be bipartite if its vertex set can be partitioned into two non-empty subsets X and Y such that each edge has one end in X and the other end in Y.



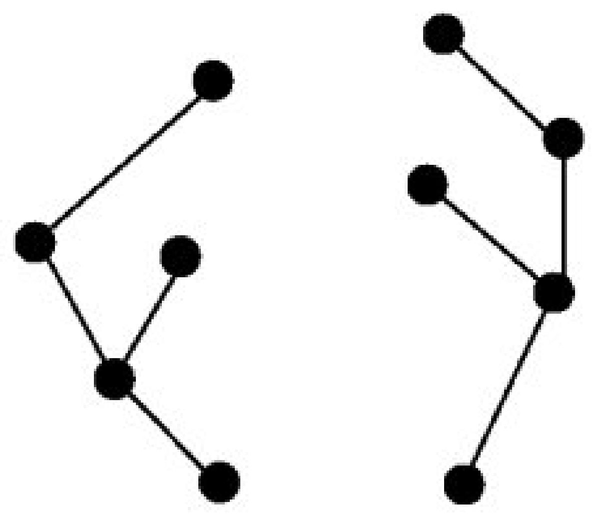
❖ SUBGRAPH: A graph H is a subgraph of a graph G if V(H) ⊆ V(G) and E(H) ⊆ E(G), where every edge in E(H) has the same end vertices as in G.

❖ TREE: A tree is a connected graph that contains no cycles.



❖ SPANNING TREE: A spanning tree of a connected graph G is a subgraph of G that is a tree and contains all the vertices of G.

❖ FOREST: A forest is an acyclic graph. A forest may be connected or disconnected. If it is connected, then it is a tree.



❖ The degree of a vertex is the number of edges incident to the vertex and is denoted by deg(v).

❖ In a directed graph, the in-degree of a vertex is the number of edges incident to the vertex and the out-degree of a vertex is the number of edges incident from the vertex.

❖ ADJACENCY MATRIX: Let G = (V,E) be a graph with vertices V = $\{v_1, v_2, \ldots, v_n\}$. The adjacency matrix of G is the $n \times n$ matrix $A(G) = [a_{ij}]$, where $a_{ij} = 1$, if v_i and v_j are adjacent and $a_{ij} = 0$, otherwise. For a simple graph, the diagonal entries are all zero because there are no loops.

- INCIDENCE MATRIX: Let $G = (V,E)$ be a graph with $V=\{v_1,v_2,\ldots,v_n\}$ as its vertex set and $E = \{e_1,e_2,\ldots,e_m\}$. The incidence matrix of G is the $n \times m$ matrix $M(G) = [m_{ij}]$, where $m_{ij} = 1$, if the vertex v_i is incident with the edge e_j and $m_{ij} = 0$, otherwise.
- WALK: A walk in a graph is a sequence of alternating vertices and edges $v_1e_1v_2e_2 \ldots\ldots v_ne_nv_{n+1}$ with $n \geq 0$. If $v_1 = v_{n+1}$ then the walk is closed. The length of the walk is the number of edges in the walk. A walk of length zero is a trivial walk.
- TRAIL: A trail is a walk with no repeated edges.
- SPAN: A subset of vertices $S \subseteq V(G)$ is said to span a subgraph H of a graph G if $V(H)=S$. If $v_1,v_2,\ldots\ldots\ldots,v_k$ are vectors in a vector space V, then the span of these vectors is the set of all linear combinations of them:

  $\text{span}\{v_1,v_2,\ldots,v_n\} = \{a_1v_1+a_2v_2+\cdots+a_nv_n : a_i \in F\}$, where F is the underlying field.
- BASIS: Let V be a vector space over a field F. A subset $B = \{v_1,v_2,\ldots,v_n\} \subseteq V$ is called a basis of V if the vectors in B are linearly independent and spans the vector space V.
- A weighted graph is a graph $G = (V, E)$ along with a function $w : E \rightarrow R$ that associates a numerical weight to each edge. If G is a weighted graph, then T is a minimal spanning tree of G if it is a spanning tree and no other spanning tree of G has smaller total weight.

CHAPTER II

MATROID THEORY

Definition 2.1. A matroid is a pair $M = (E, \mathbf{I})$ of a finite set E and a set **I** of subsets of E satisfying the following:

i. $\Phi \in \mathbf{I}$
ii. $X \in \mathbf{I}$, $Y \subseteq X \Rightarrow Y \in \mathbf{I}$, and
iii. $X, Y \in \mathbf{I}$, $|X| < |Y| \Rightarrow \exists\, e \in Y \setminus X$ such that $X \cup \{e\} \in \mathbf{I}$.

Let us say that

- X is independent if $X \in \mathbf{I}$
- X is dependent if $X \notin \mathbf{I}$.

We shall call (ii) and (iii) the hereditary and independence augmentation properties. If M is the matroid $(E,\mathbf{I})$, then M is called a matroid on E. The members of **I** are the independent sets of M, and E is the ground set of M. We shall often write $\mathbf{I}(M)$ for **I** and $E(M)$ for E, particularly when several matroids are being considered. A subset of E that is not in **I** is called dependent.

Definition 2.2. A maximal independent set is called a base.

A matroid M consists of a non-empty finite set E and a non-empty collection B of subsets of E, called bases, satisfying the following properties:

(i) no base properly contains another base;
(ii) if B_1 and B_2 are bases and if e is any element of B_1, then there is an element f of B_2 such that $(B_1 - \{e\}) \cup \{f\}$ is also a base.

(ii) is known as the exchange property. This property states that if an element is removed from B_1, then there exists an element in B_2, such that a new base, B_3, is formed when that element is added to B_1. We can use the property (ii) to show that every base in a matroid has the same number of elements.

Theorem 2.3. *Every base of a matroid has the same number of elements.*

Proof. First assume that two bases of a matroid M, B_1 and B_2, contain different number of elements, such that $|B_1| < |B_2|$. Now suppose there is some element, $\{e_1\} \in M$, such that $e_1 \in B_1$, but $e_1 \notin B_2$. If we remove $\{e_1\}$ from B_1, then by (ii), we know there is some element, $e_2 \in B_2$, but $e_2 \notin B_1$ such that $B_3 = B_1 \setminus (\{e_1\} \cup \{e_2\})$, where B_3 is a base in M. Therefore, $|B_1| = |B_3|$ but $|B_2| \neq |B_1| = |B_3|$.

If we continue the process of exchanging elements, defined by the property (ii), k number of times, then there will be no element initially in B_1 that is not in the base B_k. Therefore, for all $e \in B_k$, the element e is also in B_2, and thus $B_k \subseteq B_2$. From (i), we know that no base properly contains another base. This is a contradiction. Therefore we know that every base has the same number of elements.

Example 2.4. An example in Linear Algebra.

Consider the 5 x 8 matrix A, and its column vectors are in R^5. The set of column vectors of the matrix A are {1,2,3,4,5,6,7,8}. We will focus on the set of column vectors in a matrix as the elements of a matroid.

$$\begin{array}{cc} & \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\ \mathbf{A} = & \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

We will take the base of a matroid to be a maximal linearly independent set that spans the column space (i.e. a basis for the column space). Consider two bases of our matroid:

$$B_1 = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

$$B_2 = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

Now if we remove the second vector in B_1, then we can replace it with the second vector in B_2 to get a new base, B_3.

$$B_3 = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

For this case, (ii) is satisfied. We would find the same results if we continued this process with all possible bases of A. It is well known from Linear Algebra that no basis of A properly contains another basis.

Example 2.5. An example in Graph Theory.

We will take a base of our matroid to be a spanning tree of G.

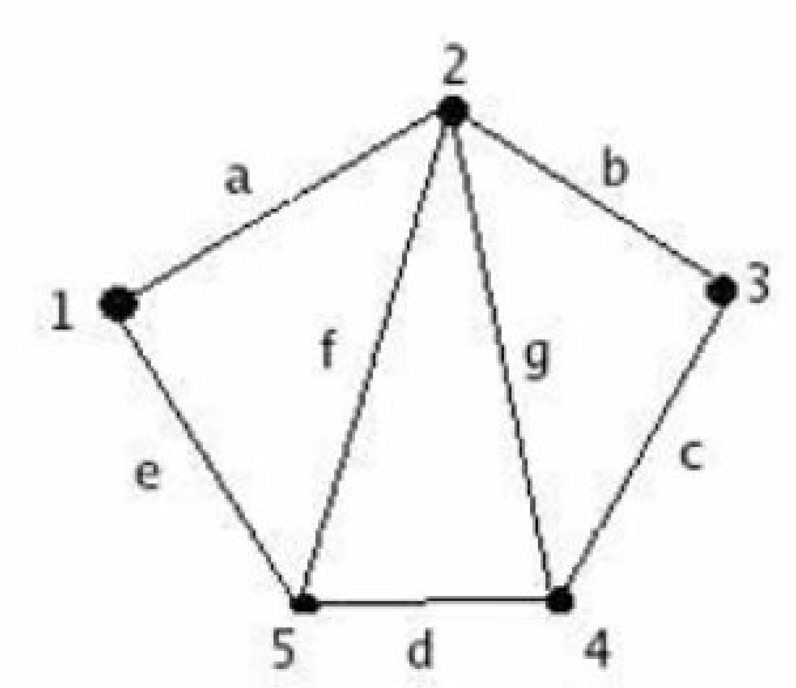


Figure 1. Graph G

Bases
$\{a, b, c, d\}$
$\{a, e, d, c\}$
$\{b, c, d, e\}$
$\{b, a, e, d\}$
$\{c, b, a, e\}$
$\{c, b, f, e\}$
$\{c, d, f, a\}$
$\{c, g, a, e\}$
$\{c, g, f, e\}$

The spanning trees of G

By observing the set of bases listed above, we can see that (i) is satisfied, because no base properly contains another base. We can now demonstrate (ii) by using this property with two bases. If we choose, $B_1 = \{a,b,c,d\}$ and $B_2 = \{c,g,a,e\}$, then we can see the spanning trees of B_1 and B_2 in figure.

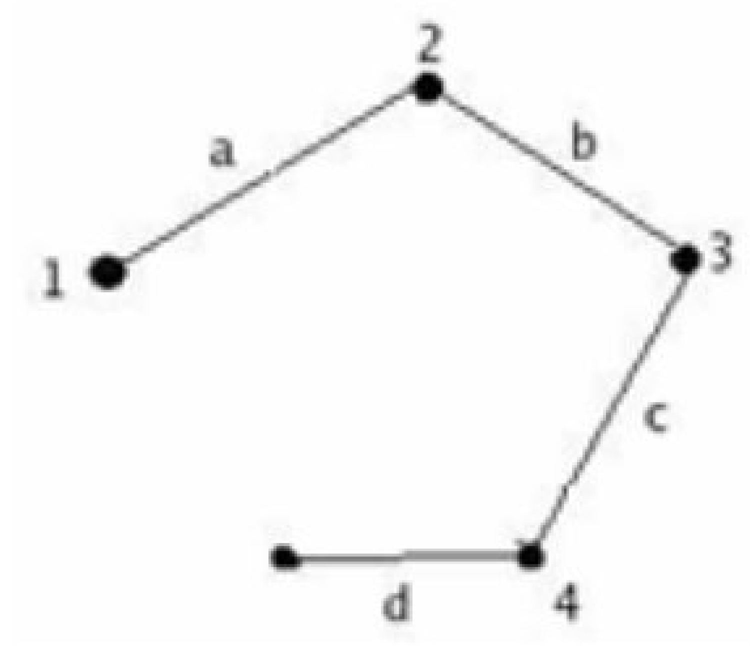


Figure 2. The spanning tree, B_1

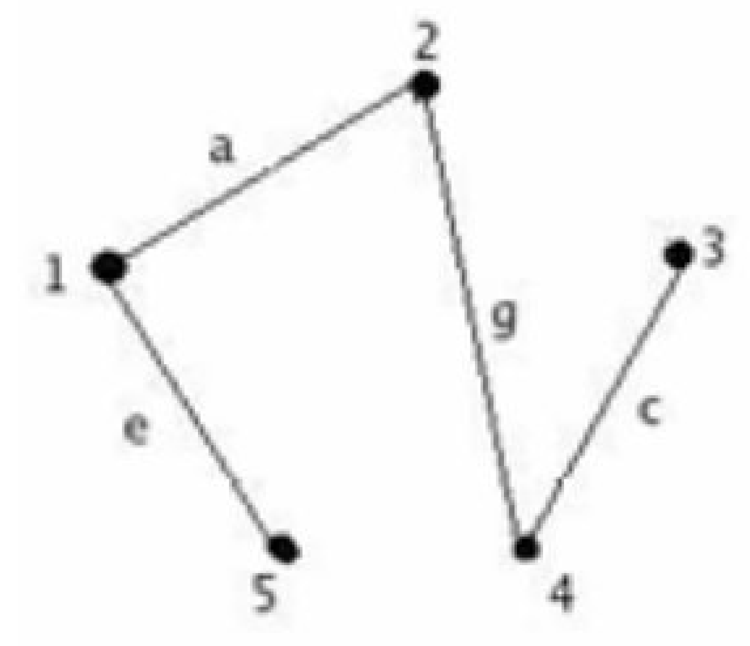


Figure 3. The spanning tree, B_2

Notice that each spanning tree has 5 vertices and 4 edges. We can demonstrate (ii) by removing an element $\{a\}$ from B_1, and then there exists an element in B_2 such that a new base is created, $B_3 = (B_1 \setminus\{a\}) \cup\{e\})$.

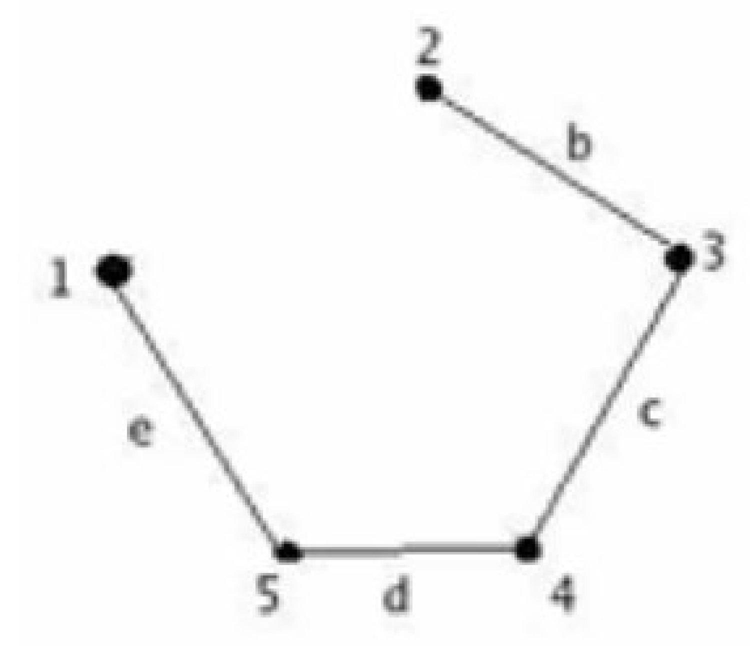


Figure 4. B_3

A similar computation works for any choice of bases.

Definition 2.6. $r_M(X) = \max \{|I| : I \subseteq X, I \in \mathbf{I}\}$ for a matroid $M = (E,\mathbf{I})$ is the rank of X. All maximal independent subsets of X have size $r_M(X)$. We call r_M the rank function of M.

A matroid consists of a non-empty finite set E and an integer-valued function r defined on the set of subset of E, satisfying:

i. $0 \leq r(A) \leq |A|$, for each subset A of E;

ii. if $A \subseteq B \subseteq E$, then $r(A) \leq r(B)$;

iii. for any $A, B \subseteq E$, $r(A \cup B) + r(A \cap B) \leq r(A) + r(B)$.

The property (i) guarantees that the rank of a subset cannot be negative, nor exceed its size. The second property guarantees that taking a superset does not decrease the rank of a set. The third property is equivalent to the exchange property that was defined in the previous section.

Definition 2.7. A loop of a matroid M is an element e of E satisfying $r(\{e\}) = 0$.

Definition 2.8. A pair of parallel elements of M is a pair $\{e, f\}$ of E that satisfy $r\{e, f\} = 1$.

Definition 2.9. A circuit is a minimal dependent set.

In graph G (Figure 1), {a,f,e}, {g,d,f}, {b,c,g} are 3 - edge circuits; {a,g,d,e}, {b,c,d,f} are 4 - edge circuits and {a,b,c,d,e} is a 5 - edge circuit.

2.10. Rank Function in Graph Theory

We can take the edges of a graph to be the elements of a matroid. For each subgraph, the rank will be the maximal number of edges in the subgraph that do not form a cycle. We can show how the rank function works in graph theory using the following example.

Let E be the set of edges of the graph in Figure 5.

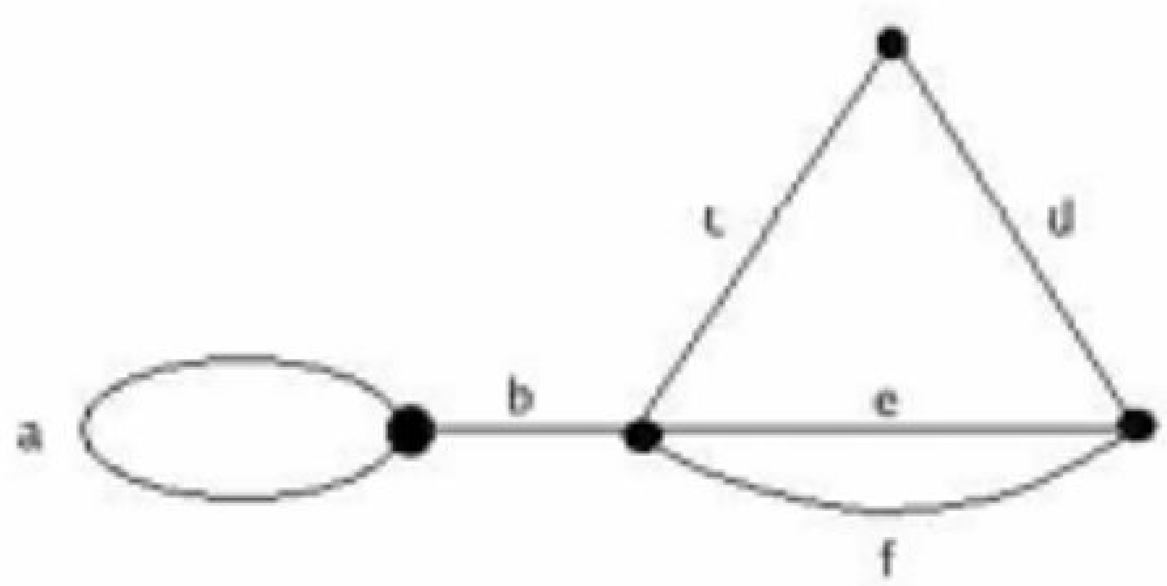


Figure 5. The set E

In Figure 6, there are no cycles and the graph is connected. Therefore $r(A) = |A| = 2$. Figure 6 is the subgraph containing $A = \{c,d\}$.

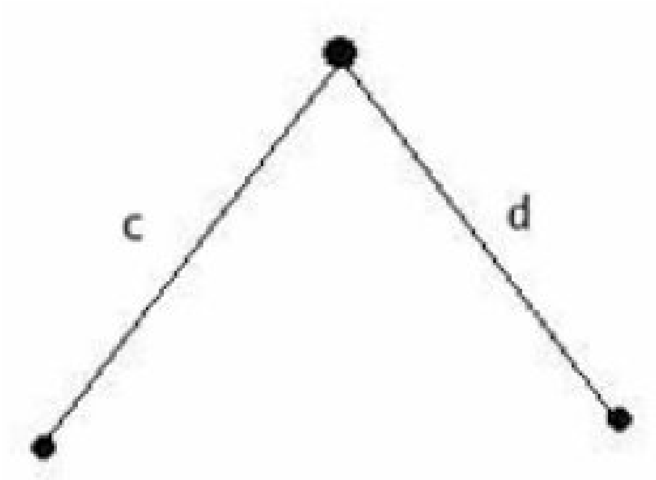


Figure 6. The subset {c, d} of E

In Figure 7, there are four elements and one cycle. The rank of B is three, because the subsets of B with the maximum number of edges, which do not contain a cycle, are {b,c,d}, {b,c,e}, and {b,e,d}. Therefore, $3 = r(B) < |B| = 4$.

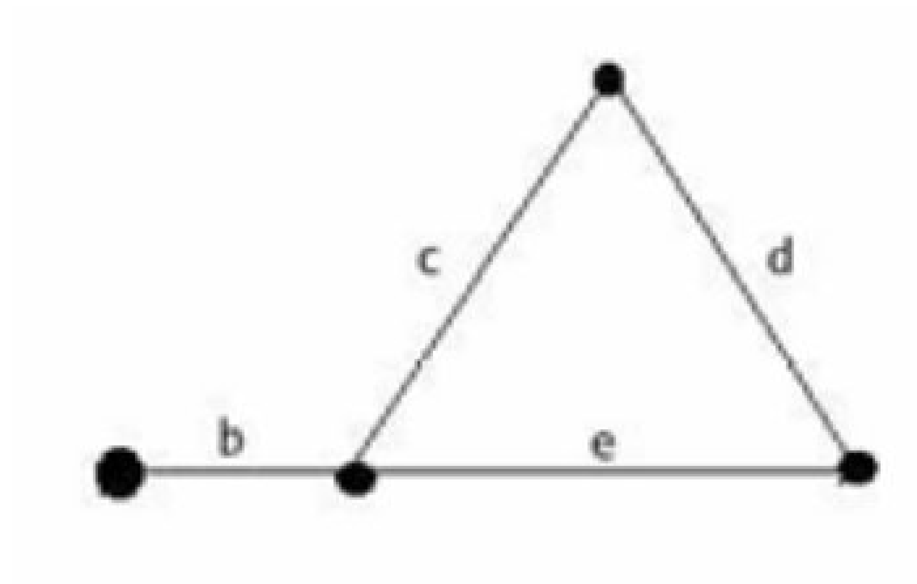


Figure 7. The subset {b,c,d,e} of E

The subset of E found in Figure 8 is a loop, with $r(C) = 0$.

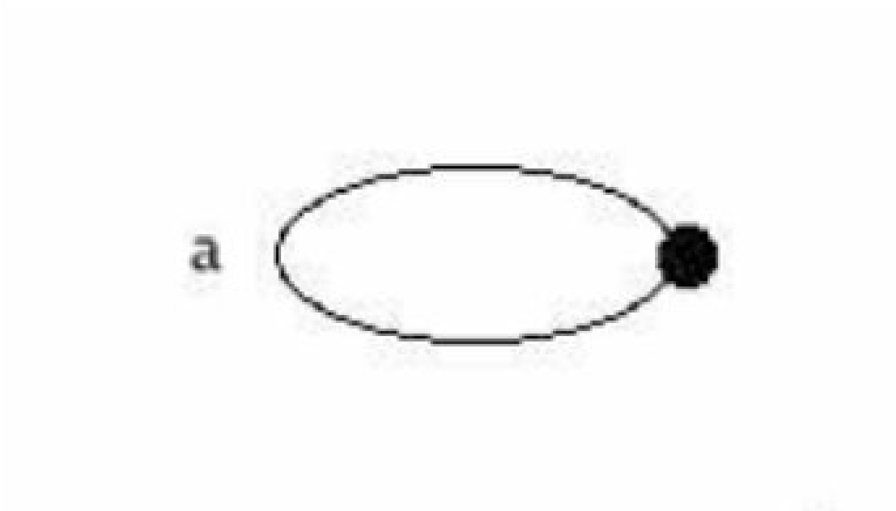


Figure 8. The subset C of E

The subset of E in Figure 9 is a set of elements that are parallel elements. Therefore, $r(D) = 1$.

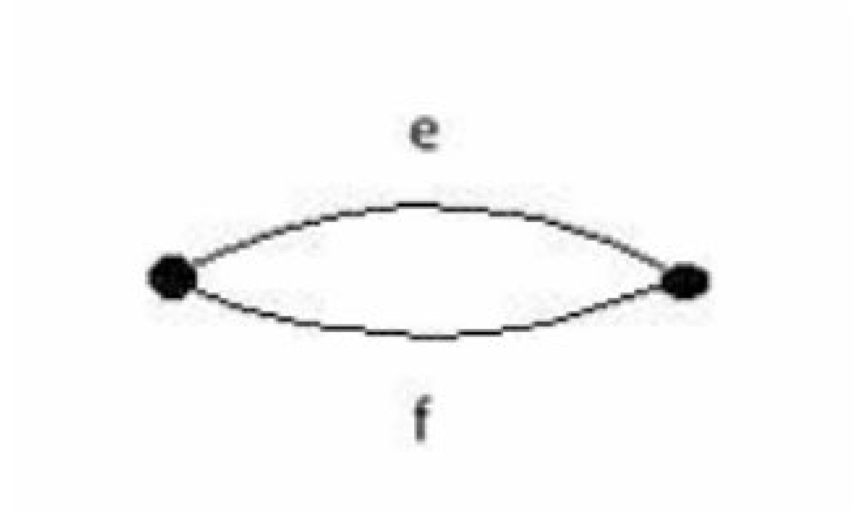


Figure 9. The subset D of E

If we take the cycle, {c,d,e}, and remove any element of the cycle, the rank of the remaining elements will always be two, as shown in Figure 6. Therefore, if we take the cycle with the remaining elements of E, we find that the rank of E is three, which means that the rank of the matroid is also three. The rank of M equals the size of a base of M.

The rank of G, in Figure 1, is 4, and because we take the set of edges of G as the elements of M, the rank of M is also r (M) = 4.

We can show an example of the property (ii) in the graph G, by considering two subsets of G, A = {a,b,e} and B = {a,b,e,f}, so that A ⊆ B ⊆ E. In this case, r (A) = r (B) = 3. However, if we let C = {a,b,d,e}, then 3 = r (A) ≤ r (B) = 4. If we continue this example with other subsets, we would come to the same conclusion.

We can demonstrate the property (iii) by using our previous example with two subsets of M being A = {a,b,e} and B = {a,b,d,e}.

$$r\,(A \cup C) + r\,(A \cap C) \leq r\,(A) + r\,(C)$$

$$r\,(\{a,b,d,e\}) + r\,(\{a,b.e\}) \leq r\,(\{a,b,d,e\}) + r\,(\{a,b,e\})$$

$$4 + 3 \leq 4 + 3$$

Therefore, property (iii) is satisfied in this case.

2.11. Rank Function in Linear Algebra.

We define rank(A) to be the size of a basis for span(A), or the dimension of the space spanned by A. Because we take the column vectors in a matrix to be the elements in our matroid, define our rank function to be the rank of each subset of M.

In our previous example, one basis of the column space of A is B_1. Because there are four column vectors in this basis, and this is a maximal linearly independent set, the rank of the matrix A is also four.

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

An example of a loop in a matrix is the zero column vector, because the space spanned by zero vector is 0 dimensional.

Definition 2.12. Two nonzero vectors, u and v are parallel elements if u=λv, for some scalar λ.

An example of a set of parallel elements in a matrix is the set {e,f} because 2e = f, the rank of the set{e,f} is one. Therefore, we say that the set{e,f} is a set of parallel elements.

$$e = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$f = \begin{bmatrix} 2 \\ 0 \\ 2 \\ 0 \\ 0 \end{bmatrix}.$$

Now we can demonstrate the properties of a matroid in terms of its rank function by examining the matrix A. We can see the property (i), by observing the set C of column vectors from the matrix A.

$$C = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

In this example, the size of C is five, while the rank of Cis four. Therefore (i) is satisfied for C. Now we will show an example of the property (ii). If we continue with this example, and take,

$$D = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

so that $D \subseteq C \subseteq A$, we can see that $3 = r(D) < r(C) = 4$. Therefore, property (ii) is satisfied in this example. From the definition of a matroid, the equation (iii) is satisfied with the two subsets, $C, D \subset E$.

$$r(C \cup D) + r(C \cap D) \leq r(C) + r(D)$$

$$4+3 \leq 4+3$$

We would come to the same conclusion if we continued to examine various subsets of M.

Definition 2.13. A subset of a matroid M is independent if it is contained in a base of a matroid.

A matroid M consists of a non-empty finite set E and a non-empty collection I of subsets of E (called independent sets) satisfying the following properties:

i. any subset of an independent set is independent;
ii. if I and J are independent sets with $|J| > |I|$, then there is an element e, contained in J but not in I, such that $I \cup \{e\}$ is independent.

Moreover, if A is an independent set, then A is contained in some base of M, which implies that $r(A) = |A|$.

2.14. Independent Sets in Graph Theory.

We will take the independent sets of a graph to be the sets of edges in a graph that do not contain a cycle. A cycle is a closed path. A forest is a graph that contains no cycles. A connected forest is a tree. We can say that the independent sets of a graph are the edge sets of the forests contained in the graph.

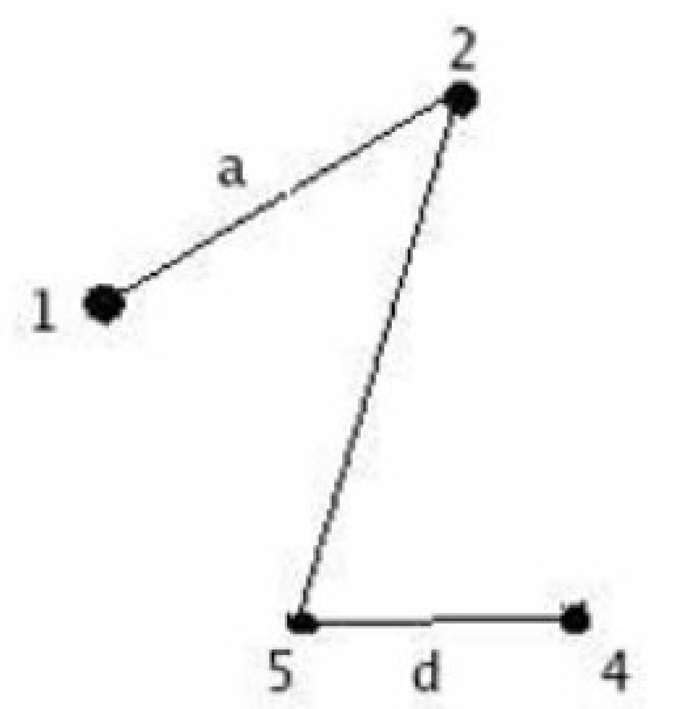


Figure 10. An example of a forest contained in G

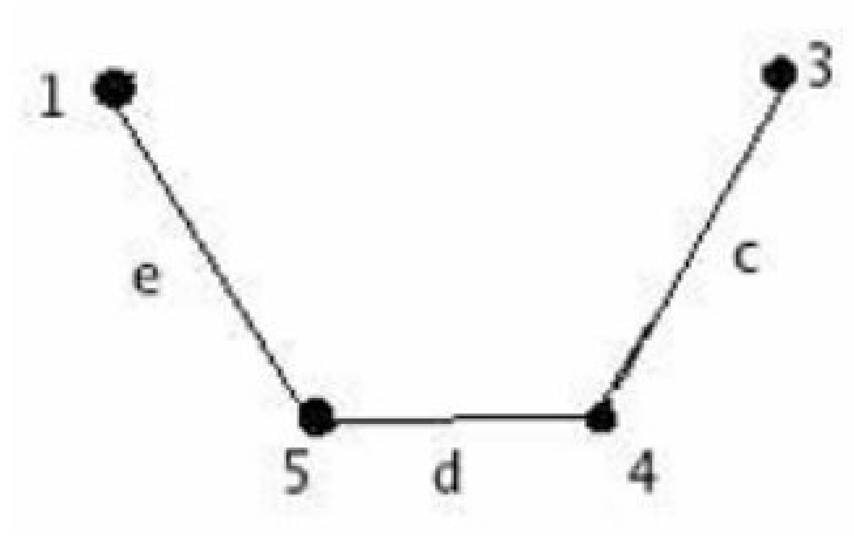


Figure 11. Another example of a forest contained in G

The first property (i), can be shown because a set is independent if it is contained within a base. Therefore independent sets must be contained within a spanning tree of a graph, which means that the rank of an independent set must be less than or equal to the rank of the graph.

Table 2.1 lists the forests contained in the graph G defined in Figure 1.

Forests of G
$\{a\}$, $\{b\}$, $\{c\}$, $\{d\}$, $\{e\}$
$\{f\}$, $\{g\}$, $\{a,b\}$, $\{b,c\}$
$\{c,d\}$, $\{d,e\}$, $\{e,f\}$, $\{f,g\}$
$\{g,a\}$, $\{a,f\}$, $\{e,f\}$, $\{d,f\}$
$\{b,f\}$, $\{b,g\}$, $\{c,g\}$, $\{d,g\}$
$\{a,b,c\}$, $\{a,b,g\}$, $\{a,e,d\}$
$\{a,f,d\}$, $\{a,g,c\}$, $\{a,g,d\}$
$\{b,c,d\}$, $\{b,g,d\}$, $\{b,f,d\}$
$\{b,f,e\}$, $\{c,d,e\}$, $\{c,g,d\}$
$\{c,d,f\}$, $\{e,f,e\}$, $\{a,f,g\}$
$\{a,b,c,d\}$, $\{a,e,d,c\}$, $\{b,c,d,e\}$
$\{b,a,e,d\}$, $\{c,b,a,e\}$, $\{c,b,f,e\}$
$\{c,d,f,a\}$, $\{c,g,a,e\}$, $\{c,g,f,e\}$

Table 2.1 Forests of G

From observing the table of forests, we can see that the forests are contained within the spanning trees, which are the bases listed in the last three rows. Now we will demonstrate why the exchange axiom for independent sets requires that two independent sets, K and L, must satisfy the inequality |K| > |L|. Suppose we let the two forests contained in G be the sets K and L_0 shown in Figures 12 and 13. Notice that |K| = |L| = 3.

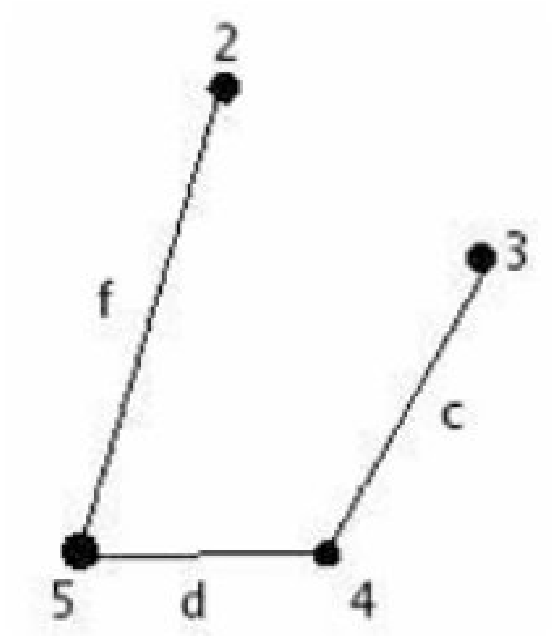


Figure 12. The forest K

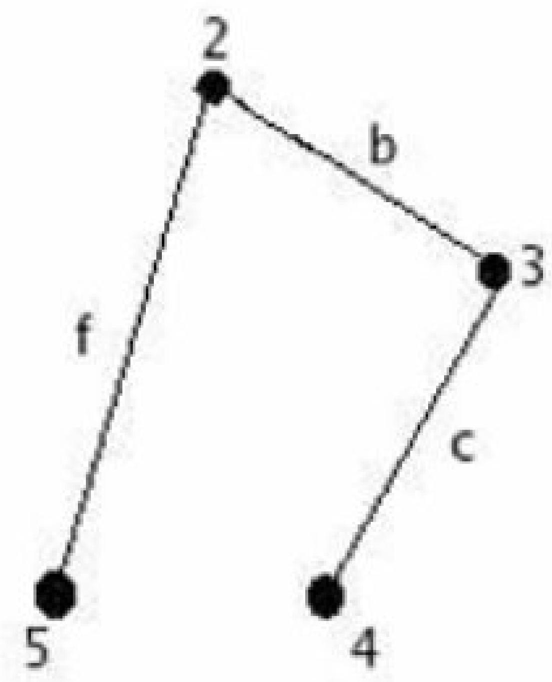


Figure 13. The forest L_0

We find that there is no element, e contained in K but not L, such that the set $L \cup \{e\}$ is independent. However, if we let $L_1 = L_0 \setminus \{c\}$, so that $3 = |K| > |L_1| = 2$, then we necessarily have an element, in this case {d}, such that $d \in K$ but not in L_1. Therefore, we find the independent set $L_1 \cup \{d\}$.

Definition 2.15. Independent sets in Linear Algebra.

An indexed set of vectors $\{v_1, v_2.., v_p\}$ in R^n is said to be linearly independent if the vector equation $x_1v_1 + x_2v_2 + ... + x_pv_p = 0$ has only the trivial solution.

The set $\{v_1..., v_p\}$ is said to be linearly dependent if there exists weights $c_1,...,c_p$, not all zero, such that $c_1v_1 + c_2v_2 + ... + c_pv_p = 0$.

We know that a base is defined as a maximal linearly independent set that spans the column space. Therefore, if we take B_1 as a base of the matrix A, namely

$$B_1 = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\},$$

Then any combination of the column vectors would create an independent set. To show that B_1 is a linearly independent set of vectors, and we can take the set of vectors, and designate them as the column vectors in the matrix, b_1

$$b_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Now we will take the column vectors as the set of vectors inB_2.

$$B_2 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

The following examples are subsets of B_2.

$$c_1 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = 0,$$

$$b_1 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + b_2 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = 0.$$

In these two examples, we find that $b_1 = b_2 = 0$ and $c_1 = c_2 = c_3 = 0$ are the only solutions. Therefore, both subsets are linearly independent.

To demonstrate (i) and (ii), we can take two independent sets of the matrix A to be K and L,

$$K = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\},$$

$$L = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\},$$

so that |K| > |L|. From our previous discussion of linear independence in a set of vectors, we can see that any subset of K or L are linearly independent.

The inequality stated in (ii) ensures that the dimension of the space spanned by K is greater than the dimension of the space spanned by L, which makes it impossible to add an element from K to L. For example, given three vectors that span a space, we can extend a different set of two vectors which span a plane to a set of three vectors which spans a space.

Now we can demonstrate the exchange property by noticing that K and L share a common element,

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix};$$

which means that we must choose one vector from the set

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\},$$

to add to L, so that Lis independent. We can see the linear independence in the sets;

$$L_1 = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\} and$$

$$L_2 = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

2.16. VERTEX – EDGE INCIDENCE MATRIX

We have seen how both graphs and matrices can be viewed as matroids. Now we will link graph theory and linear algebra by translating a graph to a unique matrix, and vice versa, using the language of matroids to motivate our discussion. The vertex-edge incidence matrix demonstrates the relationship between a matrix and a graph.

Theorem 6.1. *Let G be a graph and A_G be its vertex-edge incidence matrix. When the entries of A_G are viewed modulo (2), its vector matroid $M[A_G]$ has as its independent sets all subsets of E(G) that do not contain the edges of a cycle. Thus $M[A_G] = M(_G)$ and every graphic matroid is binary.*

The idea of a matrix being viewed mod (2), means that the entries of the matrix are either 0 or 1. Since the vertex-edge incidence matrix represents a graph, we call the graph binary. Because we have shown that a graph can be viewed as a matroid, we can say that the matroid is also binary.

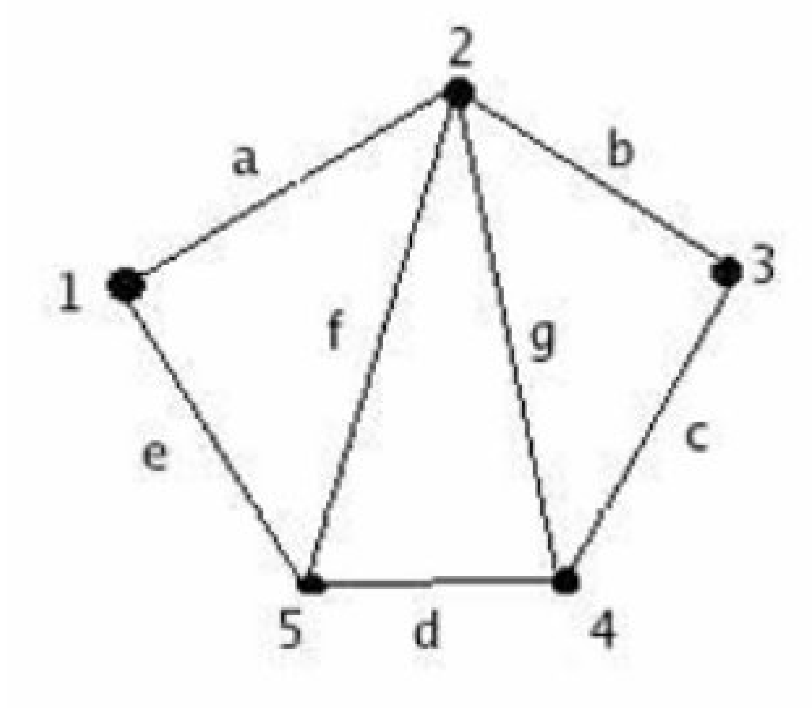


Figure 14. Graph G_3

The following is an example of a vertex-edge incidence matrix using the graph in Figure 14. If an edge and a vertex are incident on a graph, then the corresponding entry in the matrix is 1.

Otherwise, if an edge and vertex are not incident, then the corresponding entry in the matrix is zero.

$$
\begin{array}{c|ccccccc|}
 & a & b & c & d & e & f & g \\
\hline
1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
2 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\
3 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
4 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\
5 & 0 & 0 & 0 & 1 & 1 & 1 & 0
\end{array}
$$

We can also see that this set of vectors is minimally dependent. If you take any one vector from the set, the set become an independent set. The rank of the column vectors corresponding to the set {a,e,f} is also two, because any subset of the set of column vectors, containing two elements, does not contain a cycle. One base of G_3 is the set of edges {a,b,c,d}. The corresponding set of column vectors are,

$$N = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

We can see that the set of vectors is a maximal independent set, because $|N| = r\,(N) = 4$. Therefore, the set of vectors, N, is also a base.

Definition 2.16. Let $M = (E, \mathbf{I})$ be a matroid. For $X \subseteq E$, the closure (or span) of X is

$cl_M(X) = \{e \in E : r_M(X \cup \{e\}) = r_M(X)\}$.

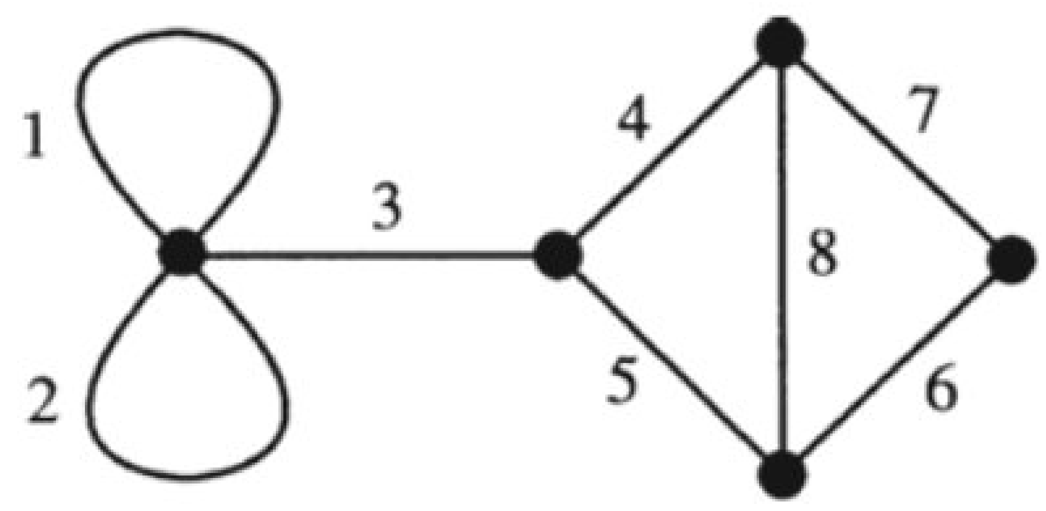


Figure 15. A graph G_1

In graph G_1, $cl(\phi) = \{1,2\}$, $cl(\{1,3,5\}) = \{1,2,3,5\}$, and $cl(\{4,5,6\}) = \{1,2,4,5,6,7,8\}$.

Definition 2.17. $X \subseteq E$ is a flat (or a closed set) if $cl(X) = X$.

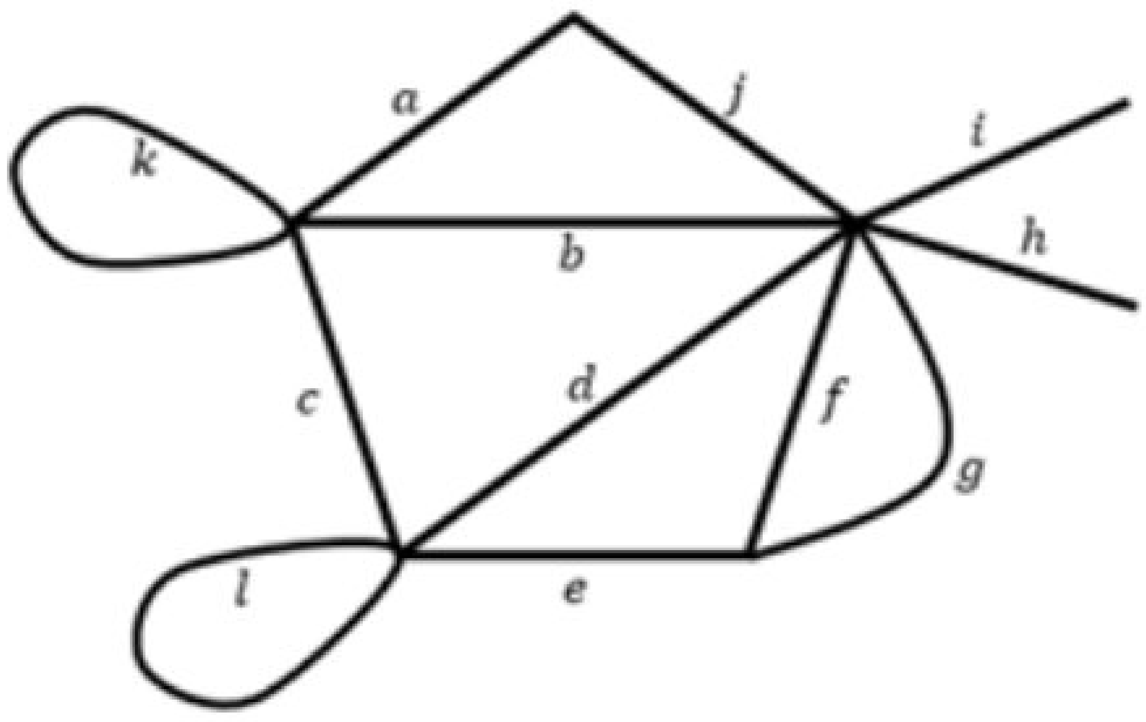


Figure 16. A graph G_2

(Flats of rank 0) = $cl(\phi)$ = (the set of all loops) = $\{k,l\}$

(Flats of rank 1) = cl(non-loop edge) = (sets of all edges joining 2 adj. vertices union all loops)

(Flats of rank 1) =$\{a,k,l\}$, $\{f,g,k,l\}$, ……

(Flats of rank 2) = $\{a,c,k,l\}$, $\{a,b,j,k,l\}$, $\{d,e,f,g,k,l\}$, ...

Definition 2.18. (Matroid via bases). Let E be a finite set and let **B** be a nonempty collection of subsets of E. We say that (E,**B**) is a matroid specified by bases if:

(i) All sets in **B** have the same finite cardinality.

(ii) (Basis exchange) For any $B_1, B_2 \in \mathbf{B}$ and any $x \in B_1 \setminus B_2$, there exists $y \in B_2 \setminus B_1$ such that $(B_1 - \{x\}) \cup \{y\} \in \mathbf{B}$.

The sets in **B** are called the bases. The independent sets are precisely the subsets of bases, and this recovers an independent-set matroid.

Definition 2.19. (Matroid via circuits). Let E be a finite set and let **C** be a collection of nonempty subsets of E. We say that (E,**C**) is a matroid specified by circuits if:

(i) No member of **C** properly contains another member of **C**.

(ii) (Circuit elimination) If $C_1, C_2 \in \mathbf{C}$ with $C_1 \neq C_2$ and $e \in C_1 \cap C_2$, then there exists a circuit $C_3 \in \mathbf{C}$ such that $C_3 \subseteq (C_1 \cup C_2) \setminus \{e\}$.

A subset $I \subseteq E$ is declared independent if it does not contain any member of C; then (E,**I**) is a matroid in the independent-set sense, and **C** is exactly the set of circuits of that matroid.

Definition 2.20. (Matroid via rank function). Let E be a finite set and let $r : 2^E \rightarrow Z_{\geq 0}$ be a function. We say that (E,r) is a matroid specified by rank if r satisfies:

(i) $0 \leq r(X) \leq |X|$ for all $X \subseteq E$.

(ii) If $X \subseteq Y \subseteq E$, then $r(X) \leq r(Y)$.

(iii) (Submodularity) For all $X, Y \subseteq E$, $r(X) + r(Y) \geq r(X \cup Y) + r(X \cap Y)$.

A set $I \subseteq E$ is declared independent if $r(I) = |I|$; with this notion of independence, (E,**I**) is a matroid and r is its rank function.

CHAPTER III

TYPES OF MATROIDS

Matroids can be classified into several types based on the mathematical structures from which they arise, while all satisfying the same axioms of independence. These types provide a unified framework for studying independence in different settings, such as graphs, vector spaces, and matrices. The most important classes include graphic (cycle) matroids, cographic (bond) matroids, vector (linear) matroids, matric matroids, and uniform matroids.

3.1. Trivial matroids

A matroid defined on any finite set E where the only independent set is the empty set ϕ.

Let E = {a,b}. The only valid independent set is **I** = {ϕ} and has rank zero. They serve as degenerate base cases in matroid.

3.2. Discrete matroids

A matroid where every possible subset of the ground set E is considered independent.

Let E = {x,y}, the independent collections are **I** ={ϕ, {x}, {y}, {x,y}}. There are no dependent sets.

3.3 Matric matroids, Vector matroids and Linear matroids

Let A be an $m \times n$ matrix over a field F. Let the ground set be the set of column indices E = {1,2,…,n}. A subset $I \subseteq E$ is independent if and only if the corresponding columns of A are linearly independent over F.

Matric matroid emphasizes that the matroid is represented by a matrix. Vector matroid emphasizes that the elements are vectors (the columns of the matrix). Linear matroid emphasizes that independence is linear independence over a field.

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

Consider columns $v_1 = (1,0)$, $v_2 = (1,1)$, $v_3 = (1,2)$. Each column is nonzero, so {1}, {2}, {3} are independent. While considering pair of columns, two vectors are independent if they are not scalar multiples of each other. Hence {1,2}, {1,3}, {2,3} are all independent. When three columns are considered, $v_3 = 2v_2 - v_1$, so {1,2,3} is dependent.

I = {ϕ, {1}, {2}, {3}, {1,2}, {1,3}, {2,3}}.

The bases are {1,2},{2,3}, {1,3}. Here the rank of the matroid is 2.

3.4. Uniform matroids

Let E be a finite set with |E| = n, and let r be an integer such that $0 \leq r \leq n$. The uniform matroid of rank r on E, denoted by $U_{r,n}$ is the matroid whose independent sets are all subsets of E having at most r elements. That is, $I = \{I \subseteq E: |I| \leq r\}$. In other words, every subset with at most r elements is independent, and every subset with more than r elements is dependent.

Consider the ground set E = {1,2,3,4}. Take the uniform matroid $U_{2,4}$ which has rank 2. Every subset containing three or more elements is dependent: {1,2,3},{1,2,4},{1,3,4},{2,3,4},{1,2,3,4}. The bases are the maximal independent sets, namely all 2-element subsets: B = {{1,2},{1,3},{1,4},{2,3},{2,4},{3,4}}.

3.5. Cycle matroids and Graphic matroids

Let G = (V,E) be a finite undirected graph, where V is the set of vertices and E is the set of edges. The cycle matroid (also called the graphic matroid) of G, denoted by M(G), is the matroid defined by M(G) = (E,**I**), where the ground set is the edge set E, and

I = {X ⊆ E:X contains no cycle in G}. Thus, a subset of edges is independent if and only if it forms a forest (an acyclic subgraph) of G.

Cycle Matroid is the term usually preferred when defining the structure directly from the graph's cycles (circuits). It emphasizes that independence means containing no cycles. Graphic matroids and cycle matroids refer to the exact same mathematical object, but they are used in slightly different contexts within matroid theory.

Imagine a graph G with 3 vertices (v_1, v_2, v_3) that form a triangle, meaning there are 3 edges such that e_1 connects v_1 to v_2, e_2 connects v_2 to v_3 and e_3 connects v_3 to v_1. Any set of 1 or 2 edges is independent because it's impossible to form a loop with fewer than 3 edges. (For example, {e_1, e_2} is independent). The set of all 3 edges is a circuit because they form a loop. If you take a subset like {e_1, e_2}, it is independent; add e_3, and it becomes a dependent circuit.

Any subset of exactly 2 edges is a base, $\{e_1, e_2\}$ is one among them. These maximal sets are the spanning trees that connect all vertices in the graph without closing a cycle.

Consider the graphic matroid $M_G = (S_G, \mathbf{I}_G)$ defined in terms of a given undirected graph $G = (V,E)$ as follows:

- The set S_G is defined to be E, the set of edges of G.
- If A is a subset of E, then A 2 G if and only if A is acyclic. That is, a set of edges A is independent if and only if the subgraph $G_A = (V,A)$ forms a forest.

Theorem 3.6. *If $G = (V, E)$ is an undirected graph, then $M_G = (S_G, \boldsymbol{I}_G)$ is a matroid.*

Proof. Clearly, $S_G = E$ is a finite set. Furthermore, G is hereditary, since a subset of a forest is a forest. Putting it another way, removing edges from an acyclic set of edges cannot create cycles. Thus, it remains to show that M_G satisfies the exchange property.

Suppose that $G_A = (V,A)$ and $G_B = (V, B)$ are forests of G and that $|B| > |A|$. That is, A and B are acyclic sets of edges, and B contains more edges than A does.

We claim that a forest $F = (V_F, E_F)$ contains exactly $|V_F| - |E_F|$ trees. To see why, suppose that F consists of t trees, where the i th tree contains v_i vertices and e_i edges. Then, we have

$| E_F | = \sum_{i=1}^{t} e_i = \sum_{i=1}^{t}(v_i - 1) = \sum_{i=1}^{t}(v_i - t) = |V_F| - t$, which implies that $t = |V_F| - |E_F|$.

Thus, forest G_A contains $|V| - |A|$ trees, and forest G_B contains $|V| - |B|$ trees.

Since forest G_B has fewer trees than forest G_A does, forest G_B must contain some tree T whose vertices are in two different trees in forest G_A. Moreover, since T is connected, it must contain an edge (u,v) such that vertices u and v are in different trees in forest G_A. Since the edge (u,v) connects vertices in two different trees in forest G_A, we can add the edge (u,v) to forest G_A without creating a cycle. Therefore, M_G satisfies the exchange property, completing the proof that M_G is a matroid.

3.6. Cographic Matroid (Bond Matroid)

Let $G = (V,E)$ be a finite undirected graph, and let M(G) be its graphic (cycle) matroid. The cographic matroid of G, denoted by M*(G), is defined as the dual matroid of the graphic (cycle) matroid M(G). That is, M*(G) = (M(G))*. The cographic matroid is also called the bond matroid because its circuits correspond to the bonds (minimal edge cuts) of the graph.

A bond is a minimal nonempty edge cut. That is, a bond is a set of edges whose removal disconnects the graph, and no proper subset of it disconnects the graph. Every cographic matroid is representable over every field (that is, it is a regular matroid).

Consider the path graph A ----- B ----- C, where e_1 connects A and B, e_2 connects B and C. Consider the edge set E = $\{e_1,e_2\}$. Removing either edge disconnects the graph. Bonds are $\{e_1\},\{e_2\}$. Therefore, the circuits of the cographic matroid are $\{\{e1\},\{e2\}\}$. The path graph is itself a spanning tree. In a tree, every edge is a bridge, and every bridge forms a bond by itself. Consequently, in the cographic matroid, every edge is a circuit of size one.

3.7. Representable matroids

A matroid M = (E,I) is representable (or linear) over a field F if there exists a matrix A over F whose columns are indexed by the elements of E such that a subset $X \subseteq E$ is independent in M if and only if the corresponding columns of A are linearly independent over F. A matroid that is representable over some field is called a representable matroid.

3.8. Regular matroids

A matroid M is called a regular matroid if it is representable over every field. Every graphic matroid is regular.

3.9. Fano matroid

The Fano matroid, denoted by F_7, is the matroid represented over the finite field F2 by the seven nonzero vectors of the three-dimensional vector space. Equivalently, it is the rank-3 matroid whose circuits are the seven lines of the Fano plane.

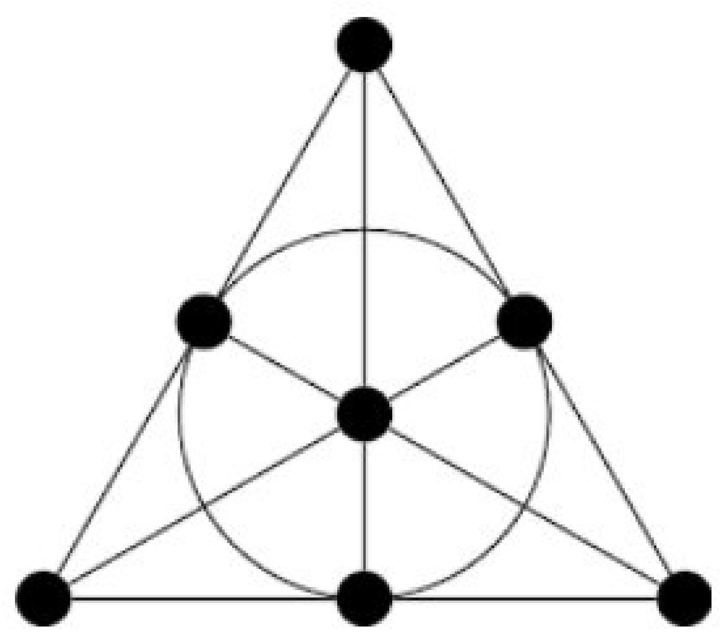


Figure 16. Fano matroid

A standard matrix representation of F_7 over F_2 is

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}$$

The seven columns of this matrix form the ground set of the matroid, and a subset of columns is independent if and only if it is linearly independent over F_2.

Consider the above matrix over F_2. Let the first four columns be

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad v_4 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Since addition is performed modulo 2, $v_1 + v_2 = v_4$. Hence, $v_1 + v_2 + v_4 = 0$, which shows that the set $\{v_1,v_2,v_4\}$ is linearly dependent. Moreover, any two of these vectors are linearly independent, so $\{v_1,v_2,v_4\}$ is a circuit of the Fano matroid. Thus, the Fano matroid is a rank-3 matroid on seven elements, whose circuits correspond to the lines of the Fano plane.

3.10. Partition matroid

Let E be a finite set partitioned into pairwise disjoint subsets $E = E_1 \cup E_2 \cup \cdots \cup E_m$, where $E_i \cap E_j = \phi$ for $i \neq j$. Let $k_1,k_2,\ldots,k_m$ be nonnegative integers satisfying $0 \leq k_i \leq |E_i|$, $i=1,2,\ldots,m$. Define $\mathbf{I} =\{I \subseteq E: |I \cap E_i| \leq k_i \text{ for every } i\}$. Then the pair $M=(E,\mathbf{I})$ is called a partition matroid.

Let $E=\{a,b,c,d,e\}$, and partition E into $E_1 = \{a,b\}, E_2 = \{c,d,e\}$. Suppose $k_1 = 1$, $k_2 = 2$. A subset is independent if it contains at most one element from E_1 and at most two elements from E_2. Hence, $\{a,c,d\},\{b,c,e\},\{c,d\}$ are independent, but $\{a,b,c\}$ and $\{a,c,d,e\}$ are not independent. Thus, the collection of independent sets is $\mathbf{I} = \{I \subseteq E: | I \cap \{a,b\} | \leq 1, | I \cap \{c,d,e\} | \leq 2\}$.

Partition matroids are heavily used in combinatorial optimization, greedy scheduling, and fair division problems. Because they model constraints where items are categorized into mutually exclusive blocks, they act as the mathematical foundation for managing resource capacities and diversity requirements.

3.11. Dual matroids and duality

The dual matroid M* of a matroid M is a matroid on E(M) having B* as the collection of bases of M* when B is the collection of bases of M. Let M=(E,I) be a matroid with ground set E, and let **B** denote the collection of bases of M. Define $\mathbf{B}^* = \{E \setminus B : B \in \mathbf{B}\}$. Then $\mathbf{B}^*$ is the collection of bases of another matroid on the same ground set E. This matroid is called the dual matroid of M and is denoted by $M^* = (E, \mathbf{B}^*)$. Thus, the bases of the dual matroid are exactly the complements of the bases of the original matroid.

Let $E = \{1,2,3\}$, and let M have bases $\mathbf{B} = \{\{1,2\},\{1,3\},\{2,3\}\}$ (This is the uniform matroid $U_{2,3}$). These are all the 2-element subsets of E. The complements of these bases are $E \setminus \{1,2\}=\{3\}$, $E \setminus \{1,3\}=\{2\}$, $E \setminus \{2,3\}=\{1\}$. Hence, the bases of the dual matroid are

$\mathbf{B}^* = \{\{1\},\{2\},\{3\}\}$. Therefore, $M^* = (E, \mathbf{B}^*)$, where every single-element subset is a basis. $(U_{r,n})^* = U_{n-r,n}$.

Important properties:

- Every matroid has a unique dual.
- The ground set of M and M* is the same.
- The bases of M* are the complements of the bases of M.
- The dual of the dual is the original matroid $(M^*)^* = M$.
- If M has rank r on a ground set of size n, then $r(M^*) = n - r(M)$.

3.12. Spanning trees as bases

Let $G = (V,E)$ be a connected graph, and let M(G) be its graphic (cycle) matroid. A basis of M(G) is the edge set of a spanning tree of G.

Thus, $\mathbf{B}(M(G)) = \{E(T) \mid T \text{ is a spanning tree of } G\}$, where E(T) denotes the edge set of the spanning tree T.

Equivalently, the bases of the graphic matroid are precisely the maximal independent sets, which correspond exactly to the spanning trees of the graph.

Consider the triangle graph with vertex set $V = \{A,B,C\}$ and edge set $E = \{e_1,e_2,e_3\}$. The graph has three spanning trees namely $\{e_1,e_2\},\{e_1,e_3\},\{e_2,e_3\}$. Therefore, the bases of the graphic matroid are $\mathbf{B}(M(G)) = \{\{e_1,e_2\},\{e_1,e_3\},\{e_2,e_3\}\}$. Each basis contains $3 - 1 = 2$ edges, as expected for a spanning tree of a connected graph with three vertices.

Equivalently, a spanning tree is a minimal connected spanning subgraph or a maximal acyclic spanning subgraph of G. Since a spanning tree on n vertices has exactly $n - 1$ edges, every spanning tree of a connected graph with n vertices contains $n-1$ edges.

Note: If G is not connected, the bases of M(G) are spanning forests, not spanning trees. Therefore, the statement "bases are exactly the spanning trees" is valid only for connected graphs.

3.13. Minimum spanning tree

Let G = (V,E) be a connected, weighted graph, where each edge $e \in E$ is assigned a weight w(e). A minimum spanning tree (MST) of G is a spanning tree whose total edge weight is minimum among all spanning trees of G. If T is a spanning tree of G, then its total weight is

$w(T) = \sum_{e \in E(T)} w(e)$, where E(T) denotes the edge set of T.

A spanning tree T is called a minimum spanning tree if $w(T) \leq w(T')$ for every spanning tree T' of G.

Consider the weighted graph with vertex set V = {A,B,C} and E = {AB,BC,AC}. The edge weights are w (AB) =2, w (BC) =1, w (AC) =3. The spanning trees are: {AB,BC} with total weight 2 + 1 = 3; {AB,AC} with total weight 2 + 3 = 5; {AC,BC} with total weight 3 + 1 = 4. Since the spanning tree {AB,BC} has the smallest total weight, it is the minimum spanning tree of the graph.

Note:

- A minimum spanning tree exists for every connected weighted graph.
- A minimum spanning tree contains exactly $|V|-1$ edges.
- If all edge weights are distinct, then the minimum spanning tree is unique.

3.14. Greedy Algorithm and Kruskal's Algorithm

Greedy Algorithm:

A greedy algorithm is an algorithmic technique for solving optimization problems by constructing a solution one step at a time. At each step, it selects the best available choice according to a specified criterion without reconsidering the choices made earlier. The algorithm continues this process until a complete solution is obtained.

A greedy algorithm is called "greedy" because it always makes the choice that appears to be the best at the current stage, hoping that these local choices together produce a globally optimal solution.

Many real-life problems require finding the best solution from a large number of possible solutions. For example, we may wish to find the shortest route between cities, construct a network with minimum cost, or schedule activities efficiently. Examining every possible solution is often impractical because the number of possibilities increases rapidly as the size of the problem grows.

The greedy algorithm provides an efficient approach for solving many such optimization problems. Instead of considering all possible solutions, it makes one decision at a time, choosing the option that seems best at that moment. Once a decision is made, it is never changed. This makes greedy algorithms simple, efficient, and easy to implement.

However, greedy algorithms do not solve every optimization problem correctly. They produce an optimal solution only for problems possessing certain mathematical properties, namely the greedy-choice property and optimal substructure.

Working Principle of the Greedy Algorithm:

The greedy algorithm follows the principle:

"At every stage, choose the best available option that satisfies the given conditions and continue until the solution is complete".

The decisions are made independently at each step. Once a choice is included in the solution, it is never removed or modified.

Characteristics of a Greedy Algorithm:

A greedy algorithm has the following characteristics:

- It builds the solution step by step.
- At each step, it selects the best available choice.
- Previously selected choices are never changed.
- It is simple to understand and implement.
- It usually requires less computation than exhaustive methods.
- It produces an optimal solution only for suitable optimization problems.

Greedy-Choice Property:

A problem satisfies the greedy-choice property if making the best local choice at each step eventually leads to an optimal solution for the entire problem.

In other words, choosing the best option at the current stage does not prevent obtaining the best overall solution.

Optimal Substructure:

A problem has optimal substructure if an optimal solution to the original problem contains optimal solutions to its smaller subproblems. This property allows the problem to be solved progressively by combining optimal solutions of smaller parts.

General Steps of a Greedy Algorithm:

The greedy algorithm can be described by the following steps.

Step 1: Begin with an empty solution.

Step 2: Identify all available choices.

Step 3: Select the best available choice according to the given criterion.

Step 4: Check whether the selected choice satisfies all constraints.

Step 5: If the choice is feasible, include it in the solution; otherwise discard it.

Step 6: Repeat Steps 2–5 until the solution is complete.

Limitations of Greedy Algorithms:

- Does not always produce an optimal solution.
- Once a decision is made, it cannot be changed.
- Depends on the greedy-choice property.
- Not suitable for every optimization problem.

Kruskal's Algorithm:

One of the most important applications of the greedy algorithm is finding a minimum spanning tree of a connected weighted graph. Among the several algorithms developed for this purpose, Kruskal's algorithm is one of the simplest and most efficient.

The algorithm was developed by Joseph Bernard Kruskal in 1956. It constructs a minimum spanning tree by repeatedly selecting the edge with the smallest weight that does not create a cycle. Since the algorithm always chooses the least expensive edge available, it is an example of a greedy algorithm.

Kruskal's algorithm is a greedy algorithm used to determine a minimum spanning tree (MST) of a connected weighted graph. It repeatedly selects the edge of minimum weight provided that the selected edge does not form a cycle with the previously chosen edges. The process continues until all the vertices are connected by exactly $n-1$ edges.

Idea of the Algorithm:

Initially every vertex is considered as a separate component. The edges are arranged in increasing order of their weights. The algorithm repeatedly selects the smallest edge. If adding the edge joins two different components, it is included. If adding the edge forms a cycle, it is discarded. The procedure stops when all the vertices become connected.

Steps of Kruskal's Algorithm:

Let $G = (V,E)$ be a connected weighted graph.

Step 1: Arrange all edges in increasing order of their weights.

Step 2: Begin with an empty set of edges.

Step 3: Select the edge having the smallest weight.

Step 4: If adding the edge creates a cycle, reject it.

Step 5: Otherwise include it in the spanning tree.

Step 6: Continue until exactly $n-1$ edges have been selected.

The resulting graph is the minimum spanning tree.

Time Complexity:

Suppose the graph contains V vertices and E edges**.** Sorting the edges requires $O(E \log E)$ time. Cycle detection using the Union–Find (Disjoint Set Union) data structure takes almost constant amortized time per operation. Therefore, the overall time complexity of Kruskal's algorithm is $O(E \log E)$.

The greedy algorithm repeatedly selects the smallest feasible edge while preserving independence. Kruskal's algorithm is exactly this greedy procedure applied to a graphic matroid. Since the greedy algorithm is guaranteed to find a minimum-weight basis in any matroid, Kruskal's algorithm always produces a minimum spanning tree for a connected weighted graph. This is one of the fundamental applications of matroid theory in graph theory and combinatorial optimization.

Comparison of Greedy algorithm and Kruskal's algorithm:

Greedy Algorithm	Kruskal's Algorithm
General optimization method	Specific greedy algorithm
Chooses the best feasible element at each step	Chooses the smallest-weight edge at each step
Maintains independence	Maintains acyclicity (independence in the graphic matroid)
Stops when a basis is obtained	Stops when a spanning tree is obtained
In a graphic matroid, the basis is a spanning tree	Produces the minimum-weight spanning tree

Table 3.1. Comparison of Greedy algorithm and Kruskal's algorithm

Let's take a simple example. Consider the connected weighted graph with edge set

$E = \{AB, AC, AD, BD, CD\}$. Their corresponding edge weights are:

Edge	Weight
AB	2
BD	3
CD	4
AD	5
AC	6

Table 3.2. Edges and their weight

Suppose we want to find a minimum-weight spanning tree. A greedy algorithm proceeds by selecting the smallest available edge at each step, provided that the selected edges remain independent, that is, they do not form a cycle.

Step 1: Select the smallest edge, AB. Selected set is $I = \{AB\}$. This set is independent.

Step 2: Choose the next smallest edge BD. Selected set is $I = \{AB, BD\}$. No cycle is formed.

Step 3: Choose CD. Selected set is $I = \{AB, BD, CD\}$. Again, no cycle is formed.

Step 4: The graph has four vertices, so a spanning tree requires $4 - 1 = 3$ edges. Hence the greedy algorithm stops.

The greedy algorithm selects $\{AB, BD, CD\}$ with total weight $2 + 3 + 4 = 9$.

Now apply Kruskal's algorithm to the same graph.

Every set of edges that contains no cycle are independent. Examples are $\phi, \{AB\}, \{AB, BD\}, \{AB, BD, CD\}$. The set $\{AB, BD, AD\}$ is dependent because it forms the cycle $A-B-D-A$.

The set $\{AB, BD, CD\}$ is a basis of the graphic matroid because it is a maximal independent set.

Another real life example can also be considered inorder to observe the correlation of Greedy algorithm in real life approaches.

Suppose a student has ₹60 to purchase books. The available books are

Book	**Price (₹)**
Mathematics	35
Physics	20
Chemistry	25
English	15

Table 3.2. Books and their price

Assume the greedy strategy is to purchase the cheapest book first.

Step 1: The cheapest book is English.

Purchase English.

Money remaining = 60 – 15 = ₹45

Step 2: Among the remaining books, the cheapest is Physics.

Purchase Physics.

Money remaining = 45 – 20 = ₹25

Step 3: Now choose Chemistry.

Money remaining = 25 – 25 = ₹0

English, Physics and Chemistry have been selected. The greedy algorithm selected the best available choice (lowest price) at every step.

3.14. Applications of matroids in Graph Theory

- **Network Design, connectivity and reliability**

  One of the most important applications of matroid theory in graph theory is the design of efficient communication and transportation networks. In a connected graph, the edge sets that do not contain cycles form the independent sets of a graphic matroid, while the spanning trees correspond to its bases. This characterization allows efficient algorithms to construct networks that connect all vertices using the minimum number of edges without creating unnecessary cycles. Consequently, matroid theory provides the mathematical foundation for designing reliable and cost-effective communication, electrical, and transportation networks.

  Matroid theory also plays an important role in studying the connectivity and reliability of graphs. The dual of a graphic matroid, known as the cographic matroid, represents the cut structure of a graph. Its circuits correspond to bonds or minimal edge cuts, which identify the smallest sets of edges whose removal disconnects the graph. This dual relationship helps in analyzing the robustness of communication, transportation, and electrical networks and in determining critical edges whose failure may disconnect the network.

- **Minimum spanning tree**

  Matroid theory provides a theoretical basis for finding minimum spanning trees in weighted graphs. Since the spanning trees of a connected graph are the bases of its graphic matroid, the problem of finding a minimum spanning tree becomes the problem of selecting a basis of minimum total weight. The greedy property of matroids guarantees that algorithms such as Kruskal's algorithm and Prim's algorithm always produce an optimal minimum spanning tree. This application is widely used in the construction of computer networks, road systems, power distribution networks, and water supply systems.

- **Optimization problems**

  Many optimization problems in graph theory can be formulated using matroids. Since matroids satisfy the greedy property, a greedy algorithm always produces an optimal solution whenever the feasible solutions form the independent sets of a matroid. This property is useful in solving problems involving edge selection, resource allocation, scheduling, and network optimization and testing graph connectivity. As a result, matroid theory provides a unified framework for studying a wide variety of combinatorial optimization problems.

In coding theory, representable matroids provide a mathematical framework for studying linear codes. The dependence relations among vectors in a vector space correspond to the dependence relations among codewords, making matroid theory useful in the design and analysis of error-correcting codes used in digital communication and data storage.

- **Computer science and engineering**

In electrical engineering, graphic and cographic matroids are applied in the analysis of electrical circuits. These concepts help in studying circuit connectivity, simplifying network structures, and identifying critical components in electrical networks. In wireless sensor networks, matroid-based optimization techniques are used to select sensors or communication links that maximize network coverage while preserving connectivity and reducing energy consumption.

Matroid theory also finds applications in transportation and power systems, where optimization techniques are employed to design efficient road networks, railway systems, electrical transmission networks, and water distribution systems. By ensuring optimal connectivity with minimal redundancy, matroid theory contributes to the efficient planning and operation of large-scale engineering networks.

CONCLUSION

6.1. CONCLUSION

Matroid theory is one of the most important branches of modern combinatorics because it provides a common mathematical framework for studying the concept of independence in different mathematical structures. Introduced by Hassler Whitney in 1935, matroid theory successfully unifies ideas from graph theory and linear algebra by identifying the essential properties shared by independent sets. Since its introduction, the theory has developed into a rich and elegant subject with deep connections to combinatorics, optimization, geometry, coding theory, and computer science. The axiomatic approach of matroid theory makes it possible to study problems arising in different mathematical disciplines using a single unified framework.

This project has presented a systematic introduction to the fundamental concepts of matroid theory. Beginning with the basic notions of graph theory, the study established the necessary background required for understanding matroids. The project then introduced the definition of a matroid and examined its principal concepts, including independent sets, bases, circuits, rank functions, closure operators, and flats. These concepts illustrate how the notion of independence can be generalized beyond vector spaces while preserving many of its important properties. The relationships among these fundamental concepts demonstrate the structural elegance and mathematical consistency of matroid theory.

Special emphasis was given to graphic matroids because they provide one of the most natural and intuitive examples of matroids. The correspondence between cycles and circuits, together with the interpretation of spanning trees as bases of a graphic matroid, demonstrates the close relationship between graph theory and matroid theory. The study of dual matroids further illustrates the principle of duality, which is one of the central ideas in matroid theory and provides valuable insight into the structural properties of matroids. These examples highlight how abstract mathematical concepts can be understood through familiar graph-theoretic structures.

The project also discussed several important applications of matroid theory in graph algorithms and optimization. In particular, the role of matroids in explaining the correctness of greedy algorithms and their application to problems such as the minimum spanning tree demonstrates the practical importance of the theory. Applications in network design, combinatorial optimization, coding theory, and computer science further show that matroid theory is not only a theoretical subject but also a valuable mathematical tool for solving real-world optimization problems. These applications emphasize the broad impact of matroid theory across several scientific and engineering disciplines. The concepts presented in this project establish a strong foundation for further study and demonstrate that matroid theory continues to be an important area of research in both pure and applied mathematics.

6.2. FUTURE SCOPE OF THE PROJECT

The present study provides only an introduction to matroid theory. Several advanced topics and research directions remain open for further exploration. Some possible areas for future study are:

- Explore the relationship between matroid theory and geometry through topics such as matroid polytopes, Bergman fans, and tropical geometry.
- Investigate matroid connectivity, minors, decomposition theorems, and excluded minor characterizations.
- Study advanced optimization techniques based on matroid intersection, matroid union, and submodular optimization.
- Study advanced classes of matroids such as representable, regular, binary, cographic, transversal, and oriented matroids.
- Examine the applications of matroid theory in graph algorithms, network design, scheduling, resource allocation, and combinatorial optimization.
- Investigate the role of matroids in coding theory, including linear codes and error-correcting codes.
- Explore computational aspects of matroids by implementing algorithms using mathematical software and programming languages.
- Study the applications of matroid theory in computer science, including algorithm design, machine learning, artificial intelligence, and data analysis.
- Investigate recent developments in algebraic combinatorics and their connections with matroid theory. Extend the study to current research problems and emerging applications of matroids in mathematics and related disciplines.

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