AN INTRODUCTION TO WAVELETS ON $\mathbb{Z}_N$

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CERTIFICATE

This is to certify that the dissertation entitled **"AN INTRODUCTION TO WAVELETS ON $\mathbb{Z}_N$"** is a record of studies carried out by **SREESHNA A S**, M.Sc. Mathematics student of the year **2024–2026** at **T.K.M. College of Arts and Science, Kollam**, under my guidance and submitted to the University of Kerala in partial fulfillment of the Degree of Master of Science in Mathematics.

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INTRODUCTION

Wavelet theory has emerged as an important area of modern mathematics due to its wide range of applications in signal processing, image compression, data analysis, and communication systems. Wavelets provide an efficient method for representing and analyzing signals at different levels of resolution. Unlike the Fourier basis, which offers excellent frequency localization but poor spatial localization, wavelets are capable of capturing both local and frequency information simultaneously.

This project presents an introduction to the construction of wavelets on the finite cyclic group $\mathbb{Z}_N$ using techniques from linear algebra and discrete Fourier analysis. The finite-dimensional setting of $\ell^2(\mathbb{Z}_N)$ provides a natural framework for understanding the fundamental concepts of wavelet theory while avoiding many of the technical difficulties encountered in the continuous case.

The study begins with the basic concepts of translation operators, convolution, discrete Fourier transforms, and orthonormal bases. Using these tools, first-stage wavelet bases are constructed through suitable low-pass and high-pass filters. The theory is then extended by an iterative process that generates multistage wavelet decompositions, leading to a family of orthonormal wavelet bases. Special attention is given to filter banks, perfect reconstruction, wavelet filter sequences, and repeated-filter constructions, which provide efficient algorithms for the decomposition and reconstruction of discrete signals.

The objective of this project is to develop a clear understanding of how wavelet bases on $\mathbb{Z}_N$ can be constructed using elementary linear algebraic methods. The finite-dimensional theory not only has independent mathematical interest but also serves as an accessible introduction to the broader theory of wavelets on $\mathbb{R}$. Through this study, we gain insight into the algebraic structure of wavelets and their usefulness in the analysis of discrete signals.

Chapter 1

PRELIMINARIES

1.1 Inner Product Spaces

Definition 1.1. *Let $\mathbb{F}$ be a field. A vector space V over $\mathbb{F}$ is a set with operations of vector addition $+$ and scalar multiplication satisfying the following properties:*

(a) (Closure for addition) For all $u, v \in V$, $u + v$ is defined and is an element of V.

(b) (Commutativity for addition) $u + v = v + u$, for all $u, v \in V$.

(c) (Associativity for addition) $u + (v + w) = (u + v) + w$, for all $u, v, w \in V$.

(d) (Existence of additive identity) There exists an element in V, denoted 0, such that $u + 0 = u$ for all $u \in V$.

(e) (Existence of additive inverse) For each $u \in V$, there exists an element in V, denoted by $-u$, such that $u + (-u) = 0$.

(f) (Closure for scalar multiplication) For all $\alpha \in \mathbb{F}$ and $u \in V$, αu is defined and is an element of V.

(g) (Behavior of the scalar multiplicative identity) $1 \cdot u = u$, for all $u \in V$, where 1 is the multiplicative identity in F.

(h) (Associativity for scalar multiplication) $\alpha\ .(\beta.\ u) = (\alpha.\beta) \cdot u$, for all α, β $\in F$ and $u \in V$.

(i) (First distributive property) $\alpha \cdot (u + v) = (\alpha \cdot u) + (\alpha \cdot v)$, for all $\alpha \in F$ and u, $v \in V$.

(j) (Second distributive property) $(\alpha + \beta) \cdot u = (\alpha \cdot u) + (\beta \cdot u)$, for all α, β $\in F$ and $u \in V$.

Definition 1.2. *Let V be a vector space over $\mathbb{C}$. A (complex) inner product is a map $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$ with the following properties:*

(a) (Additivity) $\langle u+v, w\rangle = \langle u, w\rangle + \langle v, w\rangle$ for all $u, v, w \in V$.

(b) (Scalar homogeneity) $\langle \alpha u, v\rangle = \alpha\langle u, v\rangle$ for all $\alpha \in \mathbb{C}$ and all $u, v \in V$.

(c) (Conjugate symmetry) $\langle u, v\rangle = \overline{\langle v, u\rangle}$ for all $u, v \in V$.

(d) (Positive definiteness) $\langle u, u\rangle \geq 0$ for all $u \in V$, and $\langle u, u\rangle = 0$ if and only if $u = 0$.

A vector space V with a complex inner product is called a (complex) inner product space.

Definition 1.3. *Suppose V is a complex inner product space. For $u, v \in V$, we say that u and v are orthogonal (written $u \perp v$) if $\langle u, v\rangle = 0$.*

Definition 1.4. *Suppose V is a complex inner product space. Let B be a collection of vectors in V. B is an orthogonal set if any two different elements of B are orthogonal. B is an orthonormal set if B is an orthogonal set and $\|v\| = 1$ for all $v \in B$.*

Definition 1.5. *Suppose V is a complex inner product space. An orthonormal basis for V is an orthonormal set in V that is also a basis.*

Result 1.6. *Let V be a complex inner product space with (finite) orthonormal basis R=$\{u_1, u_2, \ldots, u_n\}$.Then For any $v \in V$,*

$$v = \textstyle\sum_{j=1}^{n}\langle v, u_j\rangle u_j.$$

Definition 1.7. *Let A be an $n \times n$ matrix. A is unitary if A is invertible and $A^{-1} = A^*$.*

Result 1.8. *Let A be an $n \times n$ matrix over $\mathbb{C}$. Then the following statements are equivalent:*

(i) A is unitary.

(ii) The columns of A form an orthonormal basis for $\mathbb{C}^n$.

(iii) The rows of A form an orthonormal basis for $\mathbb{C}^n$.

(iv) A preserves inner products, that is, $\langle Az, Aw\rangle = \langle z, w\rangle$ for all $z, w \in \mathbb{C}^n$.

(v) $\|Az\| = \|z\|$ for all $z \in \mathbb{C}^n$.

Result 1.9. *Let $E = \{e_1, e_2, \ldots, e_n\}$ be the standard basis for $\mathbb{C}^n$ and suppose that $O = \{u_1, u_2, \ldots, u_n\}$ is an orthonormal basis for $\mathbb{C}^n$. Let U be the $n \times n$ matrix whose jth column is the vector u_j. Then U is unitary, U is the O-to-E change of basis matrix, and U^* is the E-to-O change of basis matrix.*

1.2 Fourier Analysis on $\mathbb{Z}_N$

Definition 1.10. *Let*

$$\mathbb{Z}_N = \{0, 1, \ldots, N-1\}.$$

The space $\ell^2(\mathbb{Z}_N)$ *is defined as*

$$\ell^2(\mathbb{Z}_N) = \{z = (z(0), z(1), \ldots, z(N-1)) : z(j) \in \mathbb{C}\}.$$

Thus each element of $\ell^2(\mathbb{Z}_N)$ *may be viewed as a complex-valued function on* $\mathbb{Z}_N$ *or equivalently as a vector in* $\mathbb{C}^N$.

Definition 1.11. *For* $z, w \in \ell^2(\mathbb{Z}_N)$*, the inner product is defined by*

$$\langle z, w \rangle = \sum_{k=0}^{N-1} z(k)\overline{w(k)}.$$

The associated norm is

$$\|z\| = \left(\sum_{k=0}^{N-1} |z(k)|^2 \right)^{1/2}.$$

For convenience, every vector $z \in \ell^2(\mathbb{Z}_N)$ is extended to all integers by requiring

$$z(j+N) = z(j), \qquad j \in \mathbb{Z}.$$

Thus elements of $\ell^2(\mathbb{Z}_N)$ are regarded as periodic functions of period N.

Definition 1.12 (Discrete Fourier Transform)**.** *Let*

$$z = (z(0), z(1), \ldots, z(N-1)) \in \ell^2(\mathbb{Z}_N)$$

.

For $m = 0, 1, \ldots, N-1$, *define*

$$\widehat{z}(m) = \sum_{n=0}^{N-1} z(n) e^{-2\pi i m n/N}. \tag{1.1}$$

The vector $\widehat{z} = (\widehat{z}(0), \widehat{z}(1), \ldots, \widehat{z}(N-1))$ *is called the* discrete Fourier transform *(DFT) of* z. *The mapping* $\widehat{\ } : \ell^2(\mathbb{Z}_N) \to \ell^2(\mathbb{Z}_N)$, *which takes* z *to* $\widehat{z}$, *is called the discrete Fourier transform.*

Result 1.13 (Fourier Inversion Formula)**.** *Let $z \in \ell^2(\mathbb{Z}_N)$. Then*

$$z(n) = \frac{1}{N}\sum_{m=0}^{N-1} \widehat{z}(m)e^{2\pi imn/N}, \qquad n = 0, 1, \ldots, N-1. \tag{1.2}$$

Definition 1.14 (Fourier Basis)**.** *For $m = 0, 1, \ldots, N-1$, define $F_m \in \ell^2(\mathbb{Z}_N)$ by*

$$F_m(n) = \frac{1}{\sqrt{N}}e^{2\pi imn/N}, \qquad n = 0, 1, \ldots, N-1. \tag{1.3}$$

Let $F = \{F_0, F_1, \ldots, F_{N-1}\}$. We call F the Fourier basis for $\ell^2(\mathbb{Z}_N)$.

Definition 1.15 (Inverse Discrete Fourier Transform)**.** *For*

$$w = (w(0), w(1), \ldots, w(N-1)) \in \ell^2(\mathbb{Z}_N),$$

define $\check{w} \in \ell^2(\mathbb{Z}_N)$ by

$$\check{w}(n) = \frac{1}{N}\sum_{m=0}^{N-1} w(m)e^{2\pi imn/N}, \qquad n = 0, 1, \ldots, N-1. \tag{1.4}$$

We set

$$\check{w} = \big(\check{w}(0), \check{w}(1), \ldots, \check{w}(N-1)\big).$$

The map $\check{} : \ell^2(\mathbb{Z}_N) \to \ell^2(\mathbb{Z}_N)$ *is called the* inverse discrete Fourier transform *(IDFT).*

Result 1.16 (Fourier Expansion Formula)**.** *Let $z \in \ell^2(\mathbb{Z}_N)$. Then*

$$z = \sum_{m=0}^{N-1} \widehat{z}(m)F_m, \tag{1.5}$$

where F_m is the m^{th} element of the Fourier basis.

Result 1.17. *The discrete Fourier transform*

$$\widehat{} : \ell^2(\mathbb{Z}_N) \to \ell^2(\mathbb{Z}_N)$$

is a linear and invertible transformation. Its inverse is the inverse discrete Fourier transform.

Definition 1.18 (Translation Operator)**.** *Let $z \in \ell^2(\mathbb{Z}_N)$ and $k \in \mathbb{Z}$. Define*

$$(R_k z)(n) = z(n-k), \qquad n \in \mathbb{Z}. \tag{1.6}$$

The vector $R_k z$ is called the translate of z by k, and R_k is called the translation operator by k.

Result 1.19. *Let $z \in \ell^2(\mathbb{Z}_N)$ and let $k \in \mathbb{Z}$. Then for every $m \in \mathbb{Z}$,*

$$\widehat{R_k z}(m) = e^{-2\pi imk/N}\widehat{z}(m). \tag{1.7}$$

The above result shows that translation in the time domain corresponds to multiplication by a complex exponential in the frequency domain.

Definition 1.20 (Translation-Invariant Linear Transformation). *A linear transformation $T : \ell^2(\mathbb{Z}_N) \to \ell^2(\mathbb{Z}_N)$ is called translation invariant if*

$$T(R_k z) = R_k T(z) \tag{1.8}$$

for every $z \in \ell^2(\mathbb{Z}_N)$ and every $k \in \mathbb{Z}$.

Definition 1.21 (Convolution). *For $z, w \in \ell^2(\mathbb{Z}_N)$, the convolution $z*w \in \ell^2(\mathbb{Z}_N)$ is defined by*

$$(z * w)(m) = \sum_{n=0}^{N-1} z(m-n)w(n), \tag{1.9}$$

for all $m \in \mathbb{Z}$.

Definition 1.22. *Define $\delta \in \ell^2(\mathbb{Z}_N)$ by*

$$\delta(n) = \begin{cases} 1, & n = 0, \\ 0, & n = 1, 2, \ldots, N-1. \end{cases} \tag{1.10}$$

The vector δ is called the unit impulse. *It is the discrete analogue of the Dirac delta function.*

Result 1.23 (Properties of Convolution). *Let $z, w \in \ell^2(\mathbb{Z}_N)$. Then*

(i)

$$z * w = w * z. \tag{1.11}$$

(ii)

$$\widehat{(z * w)}(m) = \widehat{z}(m)\widehat{w}(m), \qquad m = 0, 1, \ldots, N-1. \tag{1.12}$$

Definition 1.24 (Fast Fourier Transform). *The Fast Fourier Transform (FFT) is an efficient algorithm for computing the Discrete Fourier Transform (DFT) and its inverse.For a vector of length N, the FFT computes the DFT in approximately $N \log_2 N$ operations, compared with N^2 operations required by the direct computation of the DFT.*

The FFT makes the computation of Fourier coefficients practical for large data sets and plays an important role in signal processing, image compression, and wavelet analysis.

Result 1.25. *Suppose* $N = 2^n$ *for some* $n \in \mathbb{N}$. *Then the number of complex multiplications required by the Fast Fourier Transform satisfies*

$$\#N \leq \frac{1}{2} N \log_2 N. \tag{1.13}$$

Result 1.26. *Suppose* $z \in \ell^2(\mathbb{Z}_N)$. *Then* z *is real-valued if and only if*

$$\widehat{z}(m) = \overline{\widehat{z}(N-m)}, \qquad m \in \mathbb{Z}_N. \tag{1.14}$$

Chapter 2

CONSTRUCTION OF WAVELETS ON $\mathbb{Z}_N$: THE FIRST STAGE

2.1 Motivation for Wavelets

Throughout this chapter, we consider discrete signals represented as vectors in $\ell^2(\mathbb{Z}_N)$.

Wavelet theory was developed to overcome certain limitations of the Fourier basis in the analysis of discrete signals. Although the Fourier basis has important advantages, such as diagonalizing translation-invariant linear transformations and allowing efficient computation through the Fast Fourier Transform (FFT), its basis elements are not localized in space. Consequently, a local change in a signal can affect many Fourier coefficients, making it difficult to identify the location of important features.

In many applications, such as medical imaging, radar and sonar imaging, and video compression, it is desirable to analyze signals locally and detect where significant changes occur. This requires basis vectors that are spatially localized. At the same time, frequency localization is also important because it allows filtering, noise reduction, and efficient data compression.

A vector is said to be frequency localized if its discrete Fourier transform is concentrated in a small range of frequencies. The Fourier basis is perfectly frequency localized but not spatially localized, whereas the standard basis is perfectly spatially localized but not frequency localized. Therefore, the goal is to construct a basis whose elements are localized in both space and frequency while still permitting fast computation.

Wavelets provide such a basis and hence enable a simultaneous space-frequency analysis of signals. They combine the advantages of both spatial and frequency localization, making them highly effective for the representation and analysis of discrete signals. In this chapter, we study the construction of wavelet bases on $\mathbb{Z}_N$ and investigate their fundamental properties.

2.2 Convolution and Translation Bases

Definition 2.1 (Conjugate Reflection). *For any $w \in \ell^2(\mathbb{Z}_N)$, define $\widetilde{w} \in \ell^2(\mathbb{Z}_N)$ by*

$$\widetilde{w}(n) = \overline{w(-n)} = \overline{w(N-n)}, \qquad n \in \mathbb{Z}_N. \tag{2.1}$$

The vector $\widetilde{w}$ is called the conjugate reflection *of w.*

Result 2.2. *Let $w \in \ell^2(\mathbb{Z}_N)$. Then*

$$\widehat{\widetilde{w}}(n) = \overline{\widehat{w}(n)}, \qquad n \in \mathbb{Z}_N. \tag{2.2}$$

Recall that the translation operator R_k is defined by

$$(R_k z)(n) = z(n-k), \qquad n \in \mathbb{Z}_N. \tag{2.3}$$

Lemma 2.3. *Suppose $z, w \in \ell^2(\mathbb{Z}_N)$. For any $k \in \mathbb{Z}$,*

$$(z * \widetilde{w})(k) = \langle z, R_k w \rangle \tag{2.4}$$

and

$$(z * w)(k) = \langle z, R_k \widetilde{w} \rangle. \tag{2.5}$$

Proof. By the definition of inner product and translation,

$$\langle z, R_k w \rangle = \sum_{n=0}^{N-1} z(n)\overline{(R_k w)(n)}.$$

Since

$$(R_k w)(n) = w(n-k),$$

we get

$$\langle z, R_k w \rangle = \sum_{n=0}^{N-1} z(n)\overline{w(n-k)}.$$

By the definition of conjugate reflection,

$$\widetilde{w}(k-n) = \overline{w(n-k)}.$$

Hence

$$\langle z, R_k w\rangle = \sum_{n=0}^{N-1} z(n)\widetilde{w}(k-n).$$

By the definition of convolution,

$$\sum_{n=0}^{N-1} z(n)\widetilde{w}(k-n) = (\widetilde{w} * z)(k).$$

Since convolution is commutative,

$$(\widetilde{w} * z)(k) = (z * \widetilde{w})(k).$$

Therefore

$$(z * \widetilde{w})(k) = \langle z, R_k w\rangle.$$

The second identity follows by replacing w with $\widetilde{w}$ in the first identity. Since

$$\widetilde{\widetilde{w}} = w,$$

we obtain

$$(z * w)(k) = \langle z, R_k \widetilde{w}\rangle.$$

□

Lemma 2.3 shows that inner products with translates can be computed using convolution. Suppose $w \in \ell^2(\mathbb{Z}_N)$ is such that $B = \{R_k w\}_{k=0}^{N-1}$ is an orthonormal basis for $\ell^2(\mathbb{Z}_N)$. Then the coefficients of a vector z with respect to the basis B are $\langle z, R_k w\rangle$ (Result 1.6). By Lemma 2.3,$\langle z, R_k w\rangle = (z * \widetilde{w})(k)$.

Therefore,

$$[z]_B = z * \widetilde{w}.$$

Since convolution can be computed rapidly using the FFT, the change of basis from the Euclidean basis E to the basis B can also be computed efficiently. The next lemma gives a simple condition, in terms of the DFT of w, which characterizes all such orthonormal bases generated by translates of a single vector.

Lemma 2.4. *Let $w \in \ell^2(\mathbb{Z}_N)$. The $\{R_k w\}_{k=0}^{N-1}$ is an orthonormal basis for $\ell^2(\mathbb{Z}_N)$ if and only if $|\widehat{w}(n)| = 1$, for all $n \in (\mathbb{Z}_N)$.*

***Proof*.** Recall the Dirac delta function $\delta \in \ell^2(\mathbb{Z}_N)$ defined by

$$\delta(n) = \begin{cases} 1, & n = 0, \\ 0, & n = 1, 2, \ldots, N-1. \end{cases}$$

Since $\delta = e_0$, we have $\widehat{\delta}(n) = 1$, for all n.
By the fact that $\{R_k w\}_{k=0}^{N-1}$ is an orthonormal basis for $\ell^2(\mathbb{Z}_N)$ if and only if

$$\langle w, R_k w\rangle = \begin{cases} 1, & k = 0, \\ 0, & k = 1, 2, \ldots, N-1. \end{cases} \tag{2.6}$$

By equation(2.4) $\langle w, R_k w\rangle = (w * \widetilde{w})(k)$, so equation(2.6) is equivalent to

$$w * \widetilde{w} = \delta.$$

By Fourier inversion, equation (1.12) and equation(2.2) together with $\widehat{\widetilde{w}}(n) = \overline{\widehat{w}(n)}$, we obtain

$$1 = \widehat{\delta}(n) = (\widehat{w * \widetilde{w}})(n) = \widehat{w}(n)\widehat{\widetilde{w}}(n) = \widehat{w}(n)\overline{\widehat{w}(n)} = |\widehat{w}(n)|^2,$$

for all n

Therefore, $\quad |\widehat{w}(n)| = 1, \qquad n \in \mathbb{Z}_N.$ $\square$

Although Lemma 2.4 provides a simple characterization of orthonormal bases generated by translates of a single vector, it does not satisfy our goal of frequency localization. Indeed, if $\{R_k w\}_{k=0}^{N-1}$ is an orthonormal basis, then Lemma 2.4 implies that $|\widehat{w}(n)| = 1$ for all $n \in \mathbb{Z}_N$. Furthermore, by Result (1.19), $|\widehat{R_k w}(n)| = |\widehat{w}(n)|$, so every basis vector has the same frequency distribution. Consequently, such bases are not frequency localized. To overcome this difficulty, we modify our approach and seek two vectors u and v whose translates by even integers form an orthonormal basis. This leads to the notion of a first-stage wavelet basis.

2.3 First-Stage Wavelet Bases

Definition 2.5 (First-Stage Wavelet Basis). *Suppose N is even, say $N = 2M$ for some $M \in \mathbb{N}$. An orthonormal basis for $\ell^2(\mathbb{Z}_N)$ of the form*

$$\{R_{2k}u\}_{k=0}^{M-1} \cup \{R_{2k}v\}_{k=0}^{M-1},$$

for some $u, v \in \ell^2(\mathbb{Z}_N)$, is called a first-stage wavelet basis *for $\ell^2(\mathbb{Z}_N)$.*

The vectors u and v are called the generators of the first-stage wavelet basis. The vector u is often called the father wavelet, *while v is called the* mother wavelet.

Our goal is to determine conditions under which a pair of vectors u and v generates a first-stage wavelet basis. To obtain such a characterization, we first establish several preliminary results.

Lemma 2.6. *Suppose $M \in \mathbb{N}$, $N = 2M$, and $z \in \ell^2(\mathbb{Z}_N)$. Define $z^* \in \ell^2(\mathbb{Z}_N)$ by*

$$z^*(n) = (-1)^n z(n), \qquad n \in \mathbb{Z}_N. \tag{2.7}$$

Then

$$\widehat{z^*}(n) = \widehat{z}(n+M), \qquad n \in \mathbb{Z}_N. \tag{2.8}$$

Proof. By definition,

$$\widehat{z^*}(n) = \sum_{k=0}^{N-1} z^*(k)e^{-2\pi ikn/N} = \sum_{k=0}^{N-1}(-1)^k z(k)e^{-2\pi ikn/N}.$$

Since $(-1)^k = e^{-i\pi k}$ and $N = 2M$,

$$\widehat{z^*}(n) = \sum_{k=0}^{N-1} z(k)e^{-i\pi k}e^{-2\pi ikn/N} = \sum_{k=0}^{N-1} z(k)e^{-2\pi ik(n+M)/N} = \widehat{z}(n+M).$$

□

For any $z \in \ell^2(\mathbb{Z}_N)$, where N is even, we have

$$(z+z^*)(n) = z(n)(1+(-1)^n) = \begin{cases} 2z(n), & \text{if } n \text{ is even,} \\ 0, & \text{if } n \text{ is odd.} \end{cases} \tag{2.9}$$

Thus z^* provides a means of restricting to even values of n. This observation will be used in the proof of Lemma 2.7.

Lemma 2.7. *Suppose $M \in \mathbb{N}$, $N = 2M$, and $w \in \ell^2(\mathbb{Z}_N)$. Then $\{R_{2k}w\}_{k=0}^{M-1}$ is an orthonormal set with M elements if and only if*

$$|\widehat{w}(n)|^2 + |\widehat{w}(n+M)|^2 = 2, \quad n = 0, 1, \ldots, M-1. \tag{2.10}$$

Proof. By equation(2.4) and By the fact that, the set $\{R_{2k}w\}_{k=0}^{M-1}$ is orthonormal if and only if

$$(w * \widetilde{w})(2k) = \begin{cases} 1, & k = 0, \\ 0, & k = 1, 2, \ldots, M-1. \end{cases} \tag{2.11}$$

By equation (2.9),

$$(w * \widetilde{w} + (w * \widetilde{w})^*)(n) = \begin{cases} 2(w * \widetilde{w})(n), & n \text{ is even,} \\ 0, & n \text{ is odd.} \end{cases} \tag{2.12}$$

Hence the above condition is equivalent to

$$w * \widetilde{w} + (w * \widetilde{w})^* = 2\delta.$$

Taking Fourier transforms, we get

$$\widehat{(w * \widetilde{w})}(n) + \widehat{(w * \widetilde{w})^*}(n) = 2. \tag{2.13}$$

Now

$$\widehat{(w * \widetilde{w})}(n) = \widehat{w}(n)\widehat{\widetilde{w}}(n) = \widehat{w}(n)\overline{\widehat{w}(n)} = |\widehat{w}(n)|^2.$$

Also, by equation(2.8)

$$\widehat{(w * \widetilde{w})^*}(n) = \widehat{(w * \widetilde{w})}(n + M) = |\widehat{w}(n + M)|^2.$$

Therefore

$$|\widehat{w}(n)|^2 + |\widehat{w}(n + M)|^2 = 2.$$

Since this expression has period M, it is enough to check it for $n = 0, \ldots, M-1$. Hence $\{R_{2k}w\}_{k=0}^{M-1}$ is orthonormal if and only if

$$|\widehat{w}(n)|^2 + |\widehat{w}(n + M)|^2 = 2, \qquad n = 0, 1, \ldots, M - 1.$$

□

The condition that $\{R_{2k}w\}_{k=0}^{M-1}$ has M elements guarantees that the vectors $R_{2k}w$ are distinct for $k = 0, 1, \ldots, M - 1$.

Definition 2.8 (System Matrix). *Suppose $M \in \mathbb{N}$, $N = 2M$, and $u, v \in \ell^2(\mathbb{Z}_N)$. For $n \in \mathbb{Z}_N$, the* system matrix *of u and v is defined by*

$$A(n) = \frac{1}{\sqrt{2}} \begin{bmatrix} \widehat{u}(n) & \widehat{v}(n) \\ \widehat{u}(n + M) & \widehat{v}(n + M) \end{bmatrix}. \tag{2.14}$$

The system matrix plays a fundamental role in the study of first-stage wavelet bases. The following theorem characterizes orthonormal bases generated by the even translates of two vectors in terms of the unitarity of the system matrix.

Theorem 2.9. *Suppose $M \in \mathbb{N}$ and $N = 2M$. Let $u, v \in \ell^2(\mathbb{Z}_N)$. Then*

$$B = \{R_{2k}v\}_{k=0}^{M-1} \cup \{R_{2k}u\}_{k=0}^{M-1} = \{v, R_2v, R_4v, \ldots, R_{N-2}v, u, R_2u, R_4u, \ldots, R_{N-2}u\}.$$

is an orthonormal basis for $\ell^2(\mathbb{Z}_N)$ if and only if the system matrix

$$A(n) = \frac{1}{\sqrt{2}} \begin{bmatrix} \widehat{u}(n) & \widehat{v}(n) \\ \widehat{u}(n+M) & \widehat{v}(n+M) \end{bmatrix}$$

is unitary for each $n = 0, 1, \ldots, M-1$.

Equivalently, B is a first-stage wavelet basis for $\ell^2(\mathbb{Z}_N)$ if and only if

$$|\widehat{u}(n)|^2 + |\widehat{u}(n+M)|^2 = 2, \qquad n = 0, 1, \ldots, M-1, \tag{2.15}$$

$$|\widehat{v}(n)|^2 + |\widehat{v}(n+M)|^2 = 2, \qquad n = 0, 1, \ldots, M-1, \tag{2.16}$$

and

$$\widehat{u}(n)\overline{\widehat{v}(n)} + \widehat{u}(n+M)\overline{\widehat{v}(n+M)} = 0, \qquad n = 0, 1, \ldots, M-1. \tag{2.17}$$

Proof. Recall (Result 1.8) that a 2×2 matrix is unitary if and only if its columns form an orthonormal basis for $\mathbb{C}^2$. By Lemma 2.7, $\{R_{2k}u\}_{k=0}^{M-1}$ is orthonormal if and only if $|\widehat{u}(n)|^2 + |\widehat{u}(n+M)|^2 = 2$, that is, the first column of $A(n)$ has length 1 for every $n = 0, 1, \ldots, M-1$.

Similarly, $\{R_{2k}v\}_{k=0}^{M-1}$ is orthonormal if and only if $|\widehat{v}(n)|^2 + |\widehat{v}(n+M)|^2 = 2$, which states that the second column of $A(n)$ has length 1 for $n = 0, 1, \ldots, M-1$.

Next, we claim that

$$\langle R_{2k}u, R_{2j}v \rangle = 0 \quad \text{for all } j, k = 0, 1, \ldots, M-1 \tag{2.18}$$

if and only if $\quad \widehat{u}(n)\overline{\widehat{v}(n)} + \widehat{u}(n+M)\overline{\widehat{v}(n+M)} = 0$, for $n = 0, 1, \ldots, M-1$.

Note that the latter equation states that the columns of $A(n)$ are orthogonal. Assuming this claim momentarily, it follows that B is an orthonormal set, hence an orthonormal basis for $\ell^2(\mathbb{Z}_N)$ (because it has N elements), if and only if $A(n)$ is unitary for each $n = 0, 1, \ldots, M-1$.

To prove the equivalence of the above conditions, by equation(2.4) and by the, fact that $\langle R_{2k}u, R_{2j}v \rangle = 0$ for all $j, k = 0, 1, \ldots, M-1$ if and only if

$$(u * \widetilde{v})(2k) = \langle u, R_{2k}v \rangle = 0,$$

for all $k = 0, 1, \ldots, M-1$.

By equation (2.9) with $z = u * \widetilde{v}$, this is equivalent to

$$u * \widetilde{v} + (u * \widetilde{v})^* = 0,$$

because the values at odd indices are automatically zero.

By DFT inversion, this is equivalent to

$$\widehat{(u * \widetilde{v})}(n) + \widehat{(u * \widetilde{v})^*}(n) = 0.$$

By equation(1.12) and equation (2.2),

$$\widehat{(u * \widetilde{v})}(n) = \widehat{u}(n)\overline{\widehat{v}(n)}.$$

Hence, by equation(2.8)

$$\widehat{(u * \widetilde{v})^*}(n) = \widehat{u}(n + M)\overline{\widehat{v}(n + M)}.$$

Therefore,

$$\widehat{u}(n)\overline{\widehat{v}(n)} + \widehat{u}(n + M)\overline{\widehat{v}(n + M)} = 0.$$

Since the left-hand side is periodic with period M, it is zero for $n = 0, 1, \ldots, M-1$ if and only if it is zero for all n. Hence, the two conditions are equivalent. $\square$

Theorem 2.9 provides a practical method for constructing first-stage wavelet bases. Instead of directly verifying that

$$\{R_{2k}v\}_{k=0}^{M-1} \cup \{R_{2k}u\}_{k=0}^{M-1}$$

forms an orthonormal basis for $\ell^2(\mathbb{Z}_N)$, it is sufficient to construct vectors u and v whose system matrix $A(n)$ is unitary for all $n = 0, 1, \ldots, M - 1$.

Compared to Lemma 2.4, Theorem 2.9 provides much greater flexibility. In Lemma 2.4, the condition $|\widehat{w}(n)| = 1$ for all n prevents frequency localization. In contrast, Theorem 2.9 only requires

$$|\widehat{u}(n)|^2 + |\widehat{u}(n + M)|^2 = 2,$$

allowing the frequency components to be distributed between u and v. Consequently, one may choose u to contain primarily low-frequency components and v to contain primarily high-frequency components. For this reason, u is commonly called the *low-pass filter* (or father wavelet) and v the *high-pass filter* (or mother wavelet).

Example 2.10 (First-Stage Shannon Basis)**.** Suppose N is divisible by 4. Define $\widehat{u}, \widehat{v} \in \ell^2(\mathbb{Z}_N)$ by

$$\widehat{u}(n) = \begin{cases} \sqrt{2}, & n = 0, 1, \dots, \frac{N}{4} - 1 \text{ or } n = \frac{3N}{4}, \frac{3N}{4} + 1, \dots, N - 1, \\ 0, & n = \frac{N}{4}, \frac{N}{4} + 1, \dots, \frac{3N}{4} - 1, \end{cases}$$

and

$$\widehat{v}(n) = \begin{cases} 0, & n = 0, 1, \dots, \frac{N}{4} - 1 \text{ or } n = \frac{3N}{4}, \frac{3N}{4} + 1, \dots, N - 1, \\ \sqrt{2}, & n = \frac{N}{4}, \frac{N}{4} + 1, \dots, \frac{3N}{4} - 1. \end{cases}$$

Notice that at every n, either $\widehat{u}(n) = 0$ or $\widehat{v}(n) = 0$, so equation (2.17) holds. Also, at each n, either $\widehat{u}(n) = \sqrt{2}$ and $\widehat{u}(n + N/2) = 0$ or vice versa, so equation (2.15) holds. The same observation holds for $\widehat{v}$, so equation (2.16) also holds. Hence the system matrix $A(n)$ is unitary for all n, and therefore, by Theorem 2.9, $\{R_{2k}v\}_{k=0}^{N/2-1} \cup \{R_{2k}u\}_{k=0}^{N/2-1}$ is a first-stage wavelet basis for $\ell^2(\mathbb{Z}_N)$.

Observe that $\widehat{v}(m) = \sqrt{2}$ for the $N/2$ high frequencies

$$\frac{N}{4} \leq m \leq \frac{3N}{4} - 1,$$

and $\widehat{v}(m) = 0$ for the remaining low frequencies. Thus v contains only high-frequency components. Similarly, u contains only low-frequency components. Hence u is called the *low-pass filter* and v is called the *high-pass filter*.

2.4 Filter Banks and Perfect Reconstruction

Filter banks provide an efficient framework for the analysis and reconstruction of discrete signals. A signal is decomposed into low-frequency and high-frequency components by means of suitable filters, and the original signal is recovered from these components through a reconstruction process. The central objective is to ensure that no information is lost during decomposition and reconstruction. This property is known as *perfect reconstruction.* In this section, we introduce the basic filter bank operations and study the conditions under which perfect reconstruction is achieved.

Lemma 2.11. *Suppose $M \in \mathbb{N}$, $N = 2M$, and let $u \in \ell^2(\mathbb{Z}_N)$ be such that $\{R_{2k}u\}_{k=0}^{M-1}$ is an orthonormal set with M elements. Define $v \in \ell^2(\mathbb{Z}_N)$ by*

$$v(k) = (-1)^k \, \overline{u(1-k)}, \qquad k \in \mathbb{Z}_N. \tag{2.19}$$

Then $\{R_{2k}v\}_{k=0}^{M-1} \cup \{R_{2k}u\}_{k=0}^{M-1}$ *is a first-stage wavelet basis for* $\ell^2(\mathbb{Z}_N)$.

***Proof*.** Using the definition of v and the substitution $k = 1 - n$, we get

$$\begin{aligned}\widehat{v}(m) &= \sum_{n=0}^{N-1} v(n)e^{-2\pi imn/N} \\ &= \sum_{n=0}^{N-1} (-1)^{n-1}\overline{u(1-n)}e^{-2\pi imn/N} \\ &= e^{-2\pi im/N}\overline{\widehat{u}(m+M)}.\end{aligned}$$

Therefore,

$$\widehat{v}(m+M) = -e^{-2\pi im/N}\overline{\widehat{u}(m)}.$$

Hence

$$|\widehat{v}(m)|^2 + |\widehat{v}(m+M)|^2 = |\widehat{u}(m+M)|^2 + |\widehat{u}(m)|^2 = 2,$$

by Lemma 2.7. Thus condition (2.16) holds, and condition (2.15) holds by the hypothesis on u.

Finally, $\widehat{u}(m)\overline{\widehat{v}(m)}+\widehat{u}(m+M)\overline{\widehat{v}(m+M)} = 0$. Thus, condition (2.17) holds. Therefore, by Theorem 2.9, $\{R_{2k}v\}_{k=0}^{M-1} \cup \{R_{2k}u\}_{k=0}^{M-1}$ is a first-stage wavelet basis for $\ell^2(\mathbb{Z}_N)$. $\square$

Suppose $B = \{R_{2k}v\}_{k=0}^{M-1} \cup \{R_{2k}u\}_{k=0}^{M-1}$ is a first-stage wavelet basis. Since B is orthonormal, the coordinates of $z \in \ell^2(\mathbb{Z}_N)$ with respect to B are obtained by taking inner products with the basis elements. Hence, using Lemma 3.2, we have $\langle z, R_{2k}v\rangle = (z * \widetilde{v})(2k)$, $\langle z, R_{2k}u\rangle = (z * \widetilde{u})(2k)$.

Therefore,

$$[z]_B = \begin{bmatrix} z * \widetilde{v}(0) \\ z * \widetilde{v}(2) \\ \vdots \\ z * \widetilde{v}(N-2) \\ z * \widetilde{u}(0) \\ z * \widetilde{u}(2) \\ \vdots \\ z * \widetilde{u}(N-2) \end{bmatrix}. \tag{2.20}$$

Thus, the change of basis from the Euclidean basis to the wavelet basis can be computed by two convolutions followed by retaining only the even-indexed entries. This motivates the filter bank representation of the first-stage wavelet transform.

Definition 2.12 (Downsampling Operator)**.** *Suppose* $M \in \mathbb{N}$ *and* $N = 2M$.

Define $D : \ell^2(\mathbb{Z}_N) \to \ell^2(\mathbb{Z}_M)$ *by setting for* $z \in \ell^2(\mathbb{Z}_N)$,

$$D(z)(n) = z(2n), \qquad n = 0, 1, \ldots, M-1.$$

The operator D *is called the* downsampling *or* decimation*operator.*

In other words, if $z = (z(0), z(1), z(2), z(3), \ldots, z(N-1))$, *then* $D(z) = (z(0), z(2), z(4), \ldots, z(N-2))$.

The downsampling operator is often denoted by $\downarrow 2$ in filter bank diagrams.

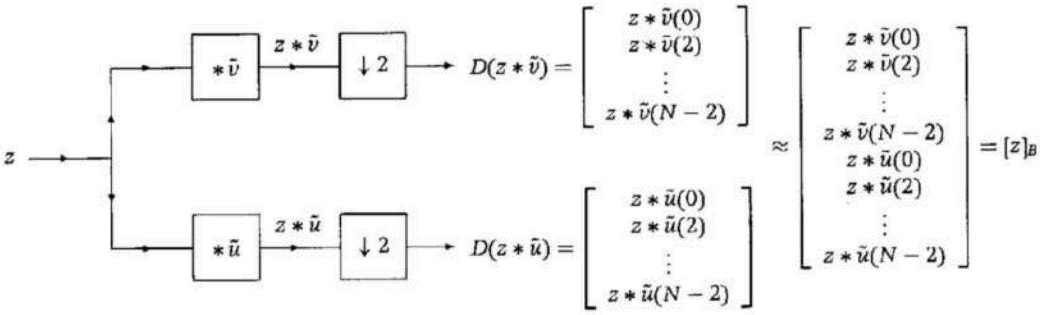

Figure 2.1: Analysis filter bank producing low-pass and high-pass components

Figure 2.1 illustrates a simple filter bank. In general, a filter bank consists of a sequence of convolutions and sampling operations used to analyze and process signals. The term *filter* refers to a convolution operator, since it can suppress or remove certain frequency components of a signal. Filter banks provide an efficient framework for computing wavelet coefficients and play an important role in signal processing applications.

Definition 2.13 (Upsampling Operator)**.** *Suppose* $M \in \mathbb{N}$ *and* $N = 2M$. *Define* $U : \ell^2(\mathbb{Z}_M) \to \ell^2(\mathbb{Z}_N)$ *by*

$$U(z)(n) = \begin{cases} z(n/2), & n \text{ is even}, \\ 0, & n \text{ is odd}. \end{cases}$$

The operator U *is called the* upsampling *operator and is denoted by* $\uparrow 2$ *in filter bank diagrams.*

For example, if $z = (2, 5, -1, i)$, then $U(z) = (2, 0, 5, 0, -1, 0, i, 0)$.
Note that if we upsample and then downsample, we got back what we started with,that is, $D(U(z)) = z$, for every $z \in \ell^2(\mathbb{Z}_M)$. However, in general, $U(D(z)) \neq z$, since the odd-indexed entries are replaced by zeros. Thus the composition $U \circ D$ has the effect of zeroing out all the odd-index values, and

hence is not the identity. By comparing $U \circ D$ with equation (2.9), we see that ,

$$U \circ D(z) = \frac{1}{2}(z + z^*). \tag{2.21}$$

To regain z from the output of the left filter bank in Figure 2.1, we follow up with a right filter bank as shown in Figure 2.2. Here $s, t \in \ell^2(\mathbb{Z}_N)$ are unknown filters.

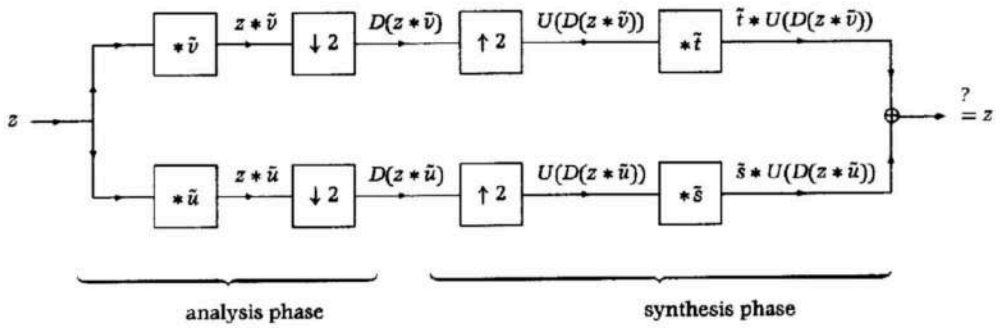

Figure 2.2: Synthesis filter bank for perfect reconstruction of the signal.

The output of the upper branch of Figure 2.2 is $\widetilde{t}*U(D(z*\widetilde{v}))$, and the output of the lower branch is $\widetilde{s}*U(D(z*\widetilde{u}))$.Lemma 2.12 gives conditions under which the sum of these outputs is always equal to the original signal z. When this happens, the filter bank is said to have *perfect reconstruction*.The following lemma provides necessary and sufficient conditions for perfect reconstruction.

Lemma 2.14. *Suppose $M \in \mathbb{N}$, $N = 2M$, and $u, v, s, t \in \ell^2(\mathbb{Z}_N)$. For $n = 0, 1, \ldots, N-1$, let $A(n)$ be the system matrix of u and v. Then Figure 2.2 has perfect reconstruction, that is,*

$$\widetilde{t} * U(D(z * \widetilde{v})) + \widetilde{s} * U(D(z * \widetilde{u})) = z$$

for all $z \in \ell^2(\mathbb{Z}_N)$, if and only if

$$A(n)\begin{bmatrix} \widehat{s}(n) \\ \widehat{t}(n) \end{bmatrix} = \begin{bmatrix} \sqrt{2} \\ 0 \end{bmatrix}, \qquad n = 0, 1, \ldots, N-1.$$

In the case that $A(n)$ is unitary, this simplifies to $\widehat{t}(n) = \overline{\widehat{v}(n)}$, and $\widehat{s}(n) = \overline{\widehat{u}(n)}$. If $A(n)$ is unitary for all n (equivalently, by Theorem 3.8, $\{R_{2k}v\}_{k=0}^{M-1} \cup \{R_{2k}u\}_{k=0}^{M-1}$ is an orthonormal basis for $\ell^2(\mathbb{Z}_N)$), then $t = \widetilde{v}$ and $s = \widetilde{u}$.

Proof. By equation (2.21),

$$U(D(z * \widetilde{v})) = \frac{1}{2}\left(z * \widetilde{v} + (z * \widetilde{v})^*\right),$$

and similarly with v replaced by u. Hence

$$U(\widehat{D(z * \widetilde{v})})(n) = \frac{1}{2}\left(\widehat{z}(n)\widehat{v}(n) + \widehat{z}(n+M)\widehat{v}(n+M)\right),$$

and similarly for u. Therefore,

$$\begin{aligned}
&\left(\widehat{\widetilde{t} * U(D(z * \widetilde{v})) + \widetilde{s} * U(D(z * \widetilde{u}))}\right)(n) \\
&= \frac{1}{2}\left(\widehat{t}(n)\widehat{v}(n) + \widehat{s}(n)\widehat{u}(n)\right)\widehat{z}(n) \\
&\quad + \frac{1}{2}\left(\widehat{t}(n)\widehat{v}(n+M) + \widehat{s}(n)\widehat{u}(n+M)\right)\widehat{z}(n+M).
\end{aligned} \tag{2.22}$$

By Fourier inversion, perfect reconstruction holds if and only if this expression in equation (2.22) is equal to $\widehat{z}(n)$ for every z and every n.We claim this holds if and only if

$$\widehat{s}(n)\widehat{u}(n) + \widehat{t}(n)\widehat{v}(n) = 2 \tag{2.23}$$

and

$$\widehat{s}(n)\widehat{u}(n+M) + \widehat{t}(n)\widehat{v}(n+M) = 0. \tag{2.24}$$

Dividing by $\sqrt{2}$ and writing these two equations in matrix form gives

$$A(n)\begin{bmatrix}\widehat{s}(n) \\ \widehat{t}(n)\end{bmatrix} = \begin{bmatrix}\sqrt{2} \\ 0\end{bmatrix}.$$

If $A(n)$ is unitary, then $A(n)^{-1} = A(n)^*$, and solving the above matrix equation gives $\widehat{s}(n) = \widehat{u}(n)$ and $\widehat{t}(n) = \widehat{v}(n)$. If $A(n)$ is unitary for all n, then by Fourier inversion and relation(2.2), $s = \widetilde{u}$ and $t = \widetilde{v}$. □

Theorem 2.14 shows how a signal can be reconstructed from its wavelet coefficients using the synthesis filter bank in Figure 2.2. When u and v generate a first-stage wavelet basis, we take $s = \widetilde{u}$ and $t = \widetilde{v}$, so that reconstruction requires only two additional convolutions and can therefore be computed efficiently.

The First-Stage Shannon Basis illustrates how a wavelet basis can separate the low-frequency and high-frequency components of a signal. In particular, the generator u acts as a low-pass filter, while v acts as a high-pass filter, providing a useful degree of frequency localization.

Next, we iterate this type of decomposition to obtain bases that capture information at different scales. This iterative procedure leads to the construction of wavelet bases and forms the subject of the next chapter.

Chapter 3

CONSTRUCTION OF WAVELETS ON $\mathbb{Z}_N$: THE ITERATION STEP

3.1 Motivation for Iteration

So far, we have constructed first-stage wavelet bases of the form

$$\{R_{2k}v\}_{k=0}^{N/2-1} \cup \{R_{2k}u\}_{k=0}^{N/2-1},$$

for $\ell^2(\mathbb{Z}_N)$. By Theorem 2.9, suitable conditions on $\widehat{u}$ and $\widehat{v}$ guarantee that this collection forms an orthonormal basis.

The First-Stage Shannon Basis demonstrates that the generators u and v can be chosen so that u contains mainly low-frequency components, while v contains mainly high-frequency components. Thus first-stage wavelet bases provide a degree of frequency localization. They also possess spatial localization and admit fast computation through filter bank algorithms.

The filter bank structure naturally suggests an iterative procedure. After decomposing a signal into low-frequency and high-frequency components, the same process can be applied again to the low-frequency part. By repeating this decomposition, we obtain a multiscale representation of the signal while preserving perfect reconstruction.

In standard wavelet analysis, the iteration is performed only on the branch corresponding to the low-pass filter. This iterative process leads to higher-stage wavelet bases, which are studied in this chapter.

3.2 The Iteration Step

To describe the iteration process, consider the filter bank diagram in Figure 16. Let u_1 and v_1 generate a first-stage wavelet basis, so that the corresponding system matrix $A(n)$ is unitary for all n. By Lemma 2.14, perfect reconstruction is obtained by using the synthesis filters u_1 and v_1.

Let $z \in \ell^2(\mathbb{Z}_N)$, where N is divisible by 4. The first-stage analysis produces the vectors $D(z * \widetilde{v}_1)$ and $D(z * \widetilde{u}_1)$. The vector $D(z * \widetilde{v}_1)$ is regarded as containing the high-frequency information and is left unchanged. The vector $D(z * \widetilde{u}_1)$, which contains the low-frequency information, is decomposed further using filters $u_2, v_2 \in \ell^2(\mathbb{Z}_{N/2})$ whose system matrix is unitary.Passing $D(z * \widetilde{u}_1)$ through the filters $\widetilde{v}_2$ and $\widetilde{u}_2$, followed by downsampling, produces the vectors $D(D(z * \widetilde{u}_1) * \widetilde{v}_2)$ and $D(D(z * \widetilde{u}_1) * \widetilde{u}_2)$.Hence, the output of the second-stage analysis consists of $D(z * \widetilde{v}_1)$, $D(D(z * \widetilde{u}_1) * \widetilde{v}_2)$, and $D(D(z * \widetilde{u}_1) * \widetilde{u}_2)$, as illustrated in Figure 3.1.

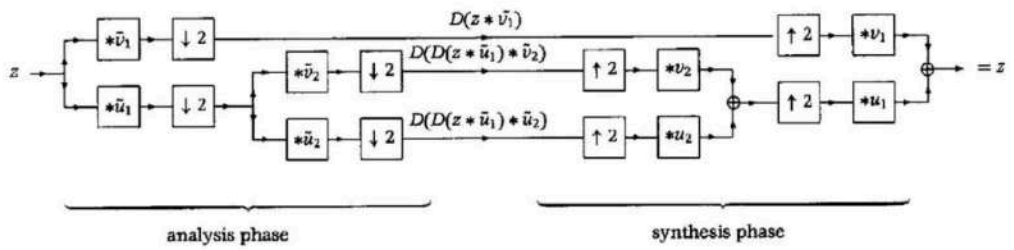

Figure 3.1: Second-stage wavelet decomposition obtained by further decomposing the low-frequency branch.

For the synthesis, or reconstruction, phase, two additional branches are introduced corresponding to the new branches in the second-stage analysis. Each branch consists of an upsampling operator followed by convolution with the filters v_2 and u_2, respectively. The outputs of these branches are then added.

By Lemma 2.14, the second-stage filter bank satisfies the perfect reconstruction property. Consequently, the overall effect of the second-stage decomposition and reconstruction is the identity. Combined with the perfect reconstruction property of the first-stage filter bank, the entire system reconstructs the original signal exactly.

If N is divisible by 2^p, this procedure can be repeated up to p times. At each stage, only the lowest branch is further decomposed using filters u_ℓ and v_ℓ satisfying the conditions of Theorem 2.9. Figure 3.2 illustrates the process at the third stage.

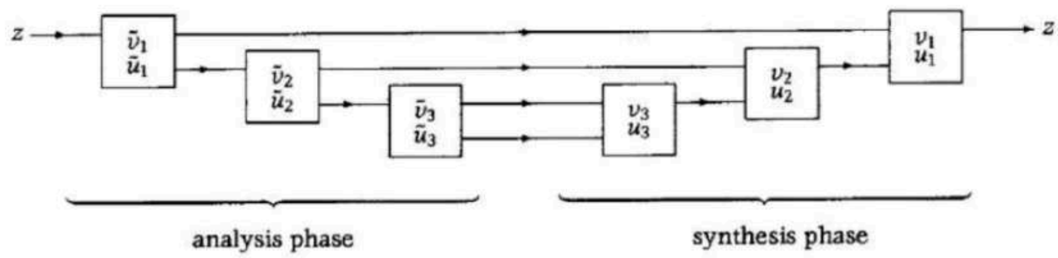

Figure 3.2: Three-stage wavelet filter bank illustrating iterative decomposition and reconstruction.

We now formalize this iterative procedure.

Definition 3.1. *Suppose N is divisible by 2^p. A p-stage* wavelet filter sequence *is a sequence of vectors $u_1, v_1, u_2, v_2, \ldots, u_p, v_p$ such that for each $\ell = 1, 2, \ldots, p$,*

$$u_\ell, v_\ell \in \ell^2(\mathbb{Z}_{N/2^{\ell-1}})$$

and the system matrix

$$A_\ell(n) = \frac{1}{\sqrt{2}} \begin{bmatrix} \widehat{u}_\ell(n) & \widehat{v}_\ell(n) \\ \widehat{u}_\ell\left(n + \frac{N}{2^\ell}\right) & \widehat{v}_\ell\left(n + \frac{N}{2^\ell}\right) \end{bmatrix} \tag{3.1}$$

is unitary for all $n = 0, 1, \ldots, (\frac{N}{2^\ell}) - 1$.

Analysis Phase

For an input $z \in \ell^2(\mathbb{Z}_N)$, define

$$x_1 = D(z * \widetilde{v}_1) \in \ell^2\left(\mathbb{Z}_{N/2}\right). \tag{3.2}$$

and

$$y_1 = D(z * \widetilde{u}_1) \in \ell^2\left(\mathbb{Z}_{N/2}\right). \tag{3.3}$$

For $\ell = 2, \ldots, p$, define recursively

$$x_\ell = D(y_{\ell-1} * \widetilde{v}_\ell) \in \ell^2\left(\mathbb{Z}_{N/2^\ell}\right), \tag{3.4}$$

and

$$y_\ell = D(y_{\ell-1} * \widetilde{u}_\ell) \in \ell^2\left(\mathbb{Z}_{N/2^\ell}\right). \tag{3.5}$$

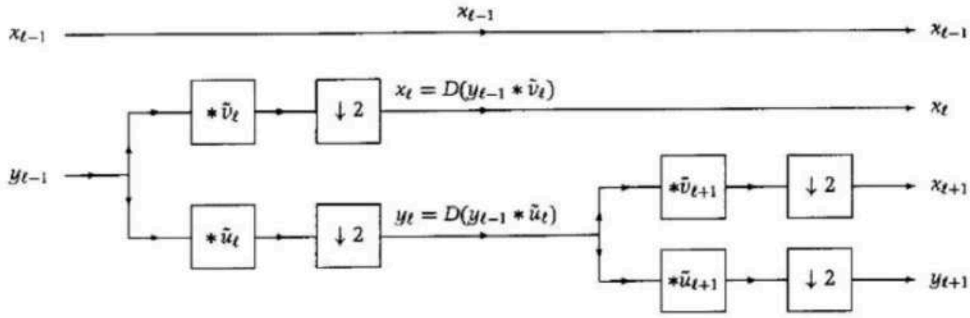

Figure 3.3: Analysis phase of a p-stage wavelet filter bank.

Note that $y_1, y_2, \ldots, y_{p-1}$ do not appear in the final output. The output of the analysis phase is $\{x_1, x_2, \ldots, x_p, y_p\}$.

The total number of coefficients is

$$\frac{N}{2} + \frac{N}{4} + \cdots + \frac{N}{2^p} + \frac{N}{2^p} = N.$$

Reconstruction Phase

The reconstruction begins with x_p and y_p. By the perfect reconstruction property,

$$(U(y_p)) * u_p + (U(x_p)) * v_p = y_{p-1}.$$

Repeating this process successively yields $y_{p-2}, y_{p-3}, \ldots, y_1$, and finally

$$z = (U(y_1)) * u_1 + (U(x_1)) * v_1.$$

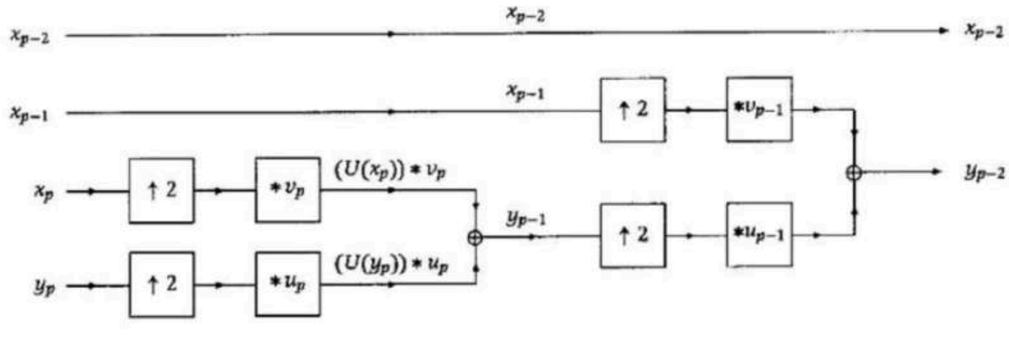

Figure 3.4: Reconstruction phase of a p-stage wavelet filter bank.

The recursive filter bank procedure can be implemented efficiently using convolutions and the Fast Fourier Transform (FFT). Although this recursive description is useful for computation, it does not directly explain how wavelet filter sequences are related to orthonormal bases of $\ell^2(\mathbb{Z}_N)$.

To obtain a clearer theoretical understanding, we introduce an equivalent nonrecursive description of the filter bank structure. This reformulation leads naturally to the construction of orthonormal wavelet bases.

3.3 Nonrecursive Description of the Filter Bank

Lemma 3.2. *Suppose N is even, say $N = 2M$, $z \in \ell^2(\mathbb{Z}_N)$, and $x, y, w \in \ell^2(\mathbb{Z}_{N/2})$. Then*

$$D(z) * w = D(z * U(w)), \tag{3.6}$$

and

$$U(x) * U(y) = U(x * y). \tag{3.7}$$

Proof. To prove (3.6), note that $w(m) = U(w)(2m)$ for each m. Hence,

$$\begin{aligned} D(z) * w(n) &= \sum_{m=0}^{(N/2)-1} D(z)(n-m)w(m) \\ &= \sum_{m=0}^{(N/2)-1} z(2n-2m)U(w)(2m) \\ &= \sum_{k=0}^{(N-1)} z(2n-k)U(w)(k) \\ &= z * U(w)(2n) \\ &= D(z * U(w))(n), \end{aligned}$$

since $U(w)(k) = 0$ whenever k is odd.

To prove (3.7), observe that $U(y)(m) = 0 \quad$ if m is odd,
and $U(y)(m) = y(m/2) \quad$ if m is even. Thus,

$$U(x) * U(y)(n) = \sum_{k=0}^{(N/2)-1} U(x)(n-2k)y(k).$$

If n is odd, then $n - 2k$ is odd for every k, so $U(x)(n-2k) = 0$. Hence,

$$U(x) * U(y)(n) = 0 = U(x * y)(n).$$

If n is even, say $n = 2\ell$, then $U(x)(n-2k) = U(x)(2\ell - 2k) = x(\ell - k)$.

$$U(x) * U(y)(n) = \sum_{k=0}^{(N/2)-1} x(\ell - k)y(k) = x * y(\ell) = (U(x * y))(n).$$

Thus

$$U(x) * U(y) = U(x * y).$$

□

Notation: For $\ell \geq 1$, let D^ℓ denote the ℓ-fold composition of the downsampling operator D with itself. Thus $D^1 = D$, and $D^\ell = D \circ D^{\ell-1}, \quad \ell > 1$.

Similarly, let U^ℓ denote the ℓ-fold composition of the upsampling operator U, that is, $U^1 = U$, and $U^\ell = U \circ U^{\ell-1}, \quad \ell > 1$.

Then, $D^\ell : \ell^2(\mathbb{Z}_N) \to \ell^2(\mathbb{Z}^\ell_{N/2})$ is given by, $D^\ell(z)(n) = z(2^\ell n)$,

whereas, $U^\ell : \ell^2(\mathbb{Z}^\ell_{N/2}) \to \ell^2(\mathbb{Z}_N)$ is given by

$$U^\ell(w)(n) = \begin{cases} w(n/2^\ell), & \text{if } 2^\ell \mid n, \\ 0, & \text{if } 2^\ell \nmid n. \end{cases}$$

Corollary 3.3. *Suppose N is divisible by 2^ℓ, $x, y, w \in \ell^2(\mathbb{Z}_{N/2^\ell})$, and $z \in \ell^2(\mathbb{Z}_N)$. Then*

$$D^\ell(z) * w = D^\ell\big(z * U^\ell(w)\big), \tag{3.8}$$

and

$$U^\ell(x * y) = U^\ell(x) * U^\ell(y). \tag{3.9}$$

Proof. The result follows by induction on ℓ. For $\ell = 1$, the two identities are exactly Lemma 3.7.

Assume the identities hold for $\ell - 1$. Then, using Lemma 3.18 and the induction hypothesis, $D^\ell(z) * w = D(D^{\ell-1}(z)) * w = D\left(D^{\ell-1}(z) * U(w)\right)$. By the induction hypothesis, $D^{\ell-1}(z) * U(w) = D^{\ell-1}\left(z * U^\ell(w)\right)$. Hence

$$D^\ell(z) * w = D^\ell\left(z * U^\ell(w)\right).$$

Similarly, using Lemma 3.7 and the induction hypothesis,

$$U^\ell(x * y) = U\left(U^{\ell-1}(x * y)\right) = U\left(U^{\ell-1}(x) * U^{\ell-1}(y)\right).$$

Applying Lemma 3.7 gives, $U^\ell(x * y) = U^\ell(x) * U^\ell(y)$. Thus both identities hold for all ℓ. $\square$

We now introduce a nonrecursive notation that is equivalent to the recursive notation of Definition 3.1.

Definition 3.4. *Suppose N is divisible by 2^p. Let $u_1, v_1, u_2, v_2, \ldots, u_p, v_p$ be vectors such that $u_\ell, v_\ell \in \ell^2(\mathbb{Z}_{N/2^{\ell-1}})$ for $\ell = 1, 2, \ldots, p$.*

Define $f_1 = v_1$, and $g_1 = u_1$. Define recursively $f_\ell, g_\ell \in \ell^2(\mathbb{Z}_N)$ for $\ell = 2, 3, \ldots, p$, by

$$f_\ell = g_{\ell-1} * U^{\ell-1}(v_\ell), \tag{3.10}$$

and

$$g_\ell = g_{\ell-1} * U^{\ell-1}(u_\ell). \tag{3.11}$$

The first few terms are

$$f_2 = u_1 * U(v_2), \qquad g_2 = u_1 * U(u_2),$$

and

$$f_3 = u_1 * U(u_2) * U^2(v_3), \qquad g_3 = u_1 * U(u_2) * U^2(u_3).$$

In general,

$$f_\ell = u_1 * U(u_2) * U^2(u_3) * \cdots * U^{\ell-2}(u_{\ell-1}) * U^{\ell-1}(v_\ell), \tag{3.12}$$

and

$$g_\ell = u_1 * U(u_2) * U^2(u_3) * \cdots * U^{\ell-2}(u_{\ell-1}) * U^{\ell-1}(u_\ell). \tag{3.13}$$

Note that all convolutions involve the filters u_j, except for the final factor in f_ℓ, which involves v_ℓ.

$$\widetilde{f}_\ell = \widetilde{g}_{\ell-1} * U^{\ell-1}(\widetilde{v}_\ell), \tag{3.14}$$

and

$$\widetilde{g}_\ell = \widetilde{g}_{\ell-1} * U^{\ell-1}(\widetilde{u}_\ell). \tag{3.15}$$

The next lemma allows us to describe the output of the analysis phase of a p-stage recursive wavelet filter bank as a collection of single (nonrecursive) convolutions. It also provides a similar description of the reconstruction phase.

Lemma 3.5. *Suppose N is divisible by 2^p, $z \in \ell^2(\mathbb{Z}_N)$, and $u_1, v_1, \ldots, u_p, v_p$ satisfy $u_\ell, v_\ell \in \ell^2(\mathbb{Z}_{N/2^{\ell-1}})$ for each $\ell = 1, 2, \ldots, p$. Define $x_1, x_2, \ldots, x_p$ and $y_1, y_2, \ldots, y_p$ by (3.2)-(3.5), and define $f_1, f_2, \ldots, f_p$ and $g_1, g_2, \ldots, g_p$ as in Definition 3.4.*

Then for $\ell = 1, 2, \ldots, p$,

$$x_\ell = D^\ell(z * \widetilde{f_\ell}), \tag{3.16}$$

and

$$y_\ell = D^\ell(z * \widetilde{g}_\ell). \tag{3.17}$$

Proof. We prove equation(3.16) and (3.17) simultaneously by induction on ℓ.

For $\ell = 1$, the results follow directly from equation (3.2)and (3.3) , and the definitions $f_1 = v_1$ and $g_1 = u_1$.

Assume that equation(3.16) and (3.17) hold for $\ell - 1$. Using equation(3.4), the induction hypothesis and equation(3.8), we obtain

$$\begin{aligned} x_\ell &= D(y_{\ell-1} * \widetilde{v}_\ell) \\ &= D\big(D^{\ell-1}(z * \widetilde{g}_{\ell-1}) * \widetilde{v}_\ell\big) \\ &= D^\ell\big(z * \widetilde{g}_{\ell-1} * U^{\ell-1}(\widetilde{v}_\ell)\big) \\ &= D^\ell(z * \widetilde{f_\ell}), \end{aligned}$$

by equation(3.14).

Similarly, using equation(3.5),

$$\begin{aligned} y_\ell &= D(y_{\ell-1} * \widetilde{u}_\ell) \\ &= D\big(D^{\ell-1}(z * \widetilde{g}_{\ell-1}) * \widetilde{u}_\ell\big) \\ &= D^\ell\big(z * \widetilde{g}_{\ell-1} * U^{\ell-1}(\widetilde{u}_\ell)\big) \\ &= D^\ell(z * \widetilde{g}_\ell), \end{aligned}$$

by equation(3.15).

Hence equation(3.16) and (3.17) hold for all $\ell = 1, 2, \ldots, p$. $\square$

Lemma 3.6. *Suppose* N *is divisible by* 2^p. *Consider a* p*-stage filter bank sequence* $u_1, v_1, \ldots, u_p, v_p$ *as in Definition 3.1 (except that we do not require the system matrix in equation(3.1) to be unitary for this result). Define* $f_1, \ldots, f_p, g_p$ *as in Definition 3.4. If the input to the* ℓ*th branch* $(1 \leq \ell \leq p)$ *of the reconstruction phase (i.e., the branch for which the next operation is convolution with* v_ℓ*) is* x_ℓ*, and all other inputs are zero, then the output of the reconstruction phase is*

$$f_\ell * U^\ell(x_\ell).$$

If the input to the final branch (for which the next operation is convolution with u_p*) is* y_p*, and all other inputs are zero, then the output of the reconstruction phase is*

$$g_p * U^p(y_p).$$

Thus the full recursive pth-stage wavelet filter sequence can be represented by the nonrecursive structure shown in Figure 3.5.

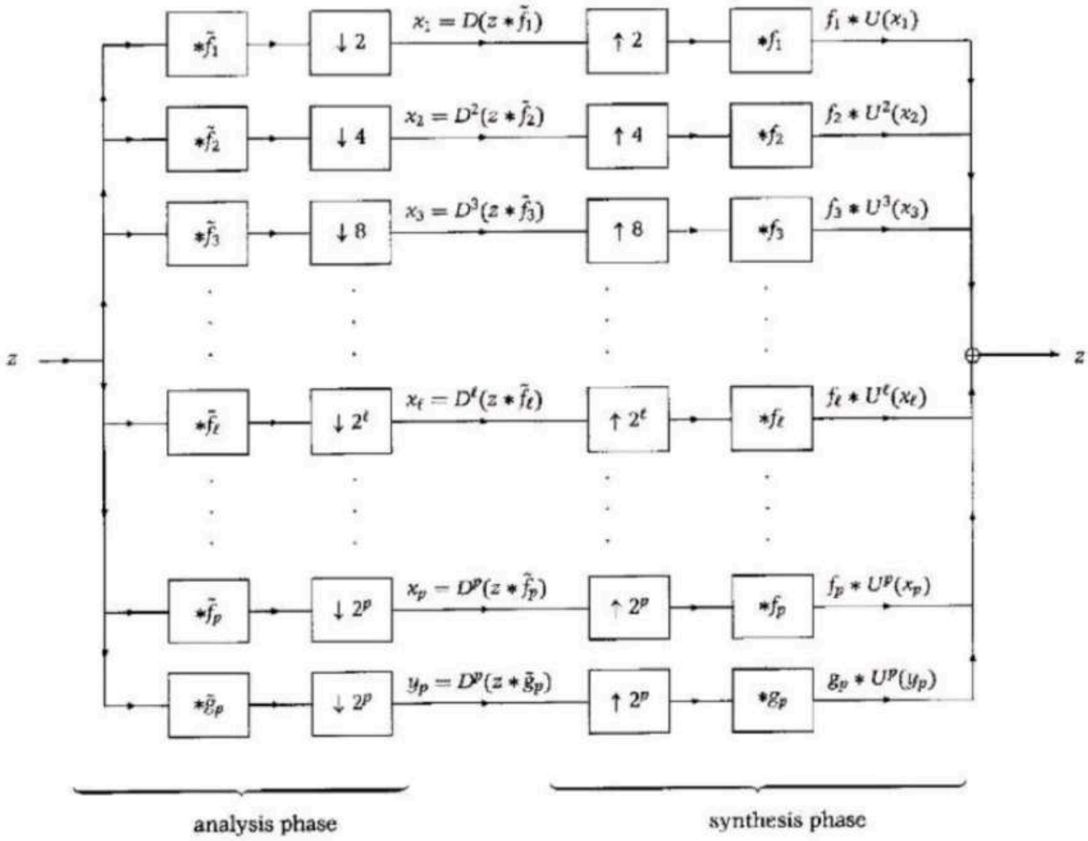

Figure 3.5:

Recall that the output of the analysis phase of the filter bank is the collection of vectors $x_1, x_2, \ldots, x_p, y_p$. By Lemma 3.5, for each $\ell = 1, 2, \ldots, p$,

$$x_\ell(k) = D^\ell(z * \widetilde{f}_\ell)(k) = z * \widetilde{f}_\ell(2^\ell k) = \langle z, R_{2^\ell k} f_\ell \rangle, \tag{3.18}$$

for $k = 0, 1, \ldots, (N/2^\ell) - 1$.

Similarly,

$$y_p(k) = D^p(z * \widetilde{g}_p)(k) = z * \widetilde{g}_p(2^p k) = \langle z, R_{2^p k} g_p \rangle, \tag{3.19}$$

for $k = 0, 1, \ldots, (N/2^p) - 1$. Since the total number of components of $x_1, x_2, \ldots, x_p, y_p$ is N, these quantities are natural candidates for the coefficients of z with respect to an orthonormal basis. This observation leads to the construction of wavelet bases, which is developed in the next section.

3.4 Construction of Wavelet Bases

Definition 3.7. *Suppose N is divisible by 2^p, where p is a positive integer. Let B be a set of the form*

$$\{R_{2k}f_1\}_{k=0}^{N/2-1} \cup \{R_{4k}f_2\}_{k=0}^{N/4-1} \cup \cdots \cup \{R_{2^pk}f_p\}_{k=0}^{N/2^p-1} \cup \{R_{2^pk}g_p\}_{k=0}^{N/2^p-1},$$

for some $f_1, f_2, \ldots, f_p, g_p \in \ell^2(\mathbb{Z}_N)$. If B forms an orthonormal basis for $\ell^2(\mathbb{Z}_N)$, then B is called a p-stage wavelet basis *for $\ell^2(\mathbb{Z}_N)$. The vectors $f_1, f_2, \ldots, f_p, g_p$ are said to* generate B.

Our goal is to show that the vectors $f_1, f_2, \ldots, f_p, g_p$ obtained from Definitions 3.1 and 3.4 generate a p-stage wavelet basis. The key step is contained in the following lemma.

Lemma 3.8. *Suppose N is divisible by 2^ℓ, $g_{\ell-1} \in \ell^2(\mathbb{Z}_N)$, and the set*

$$\{R_{2^{\ell-1}k}g_{\ell-1}\}_{k=0}^{N/2^{\ell-1}-1} \tag{3.20}$$

is orthonormal with $N/2^{\ell-1}$ elements.

Suppose $u_\ell, v_\ell \in \ell^2(\mathbb{Z}_{N/2^{\ell-1}})$ and the system matrix $A_\ell(n)$ in (3.1) is unitary for $n = 0, 1, \ldots, (N/2^\ell) - 1$.

Define

$$f_\ell = g_{\ell-1} * U^{\ell-1}(v_\ell), \qquad g_\ell = g_{\ell-1} * U^{\ell-1}(u_\ell).$$

Then

$$\{R_{2^\ell k}f_\ell\}_{k=0}^{N/2^\ell-1} \cup \{R_{2^\ell k}g_\ell\}_{k=0}^{N/2^\ell-1} \tag{3.21}$$

is an orthonormal set with $N/2^{\ell-1}$ elements.

Proof. Since the system matrix $A_\ell(n)$ is unitary, Theorem 32.9 implies that $\{R_{2k}v_\ell\}_{k=0}^{N/2^\ell-1} \cup \{R_{2k}u_\ell\}_{k=0}^{N/2^\ell-1}$ is an orthonormal basis for $\ell^2(\mathbb{Z}_{N/2^{\ell-1}})$. Hence

$$(v_\ell * \widetilde{v}_\ell)(2k) = \begin{cases} 1, & k = 0, \\ 0, & k = 0, 1, \ldots, (N/2^\ell) - 1 \end{cases} \tag{3.22}$$

$$(u_\ell * \widetilde{u}_\ell)(2k) = \begin{cases} 1, & k = 0, \\ 0, & k = 0, 1, \ldots, (N/2^\ell) - 1, \end{cases} \tag{3.23}$$

and

$$(v_\ell * \widetilde{u}_\ell)(2k) = 0.$$

Using the definitions

$$f_\ell = g_{\ell-1} * U^{\ell-1}(v_\ell), \qquad g_\ell = g_{\ell-1} * U^{\ell-1}(u_\ell),$$

together with equations (3.9), equation(3.12), and the orthonormality of $\{R_{2^{\ell-1}k}g_{\ell-1}\}$, one obtains

$$\langle f_\ell, R_{2^\ell k} f_\ell\rangle = (v_\ell * \widetilde{v}_\ell)(2k),$$

$$\langle g_\ell, R_{2^\ell k} g_\ell\rangle = (u_\ell * \widetilde{u}_\ell)(2k),$$

and

$$\langle f_\ell, R_{2^\ell k} g_\ell\rangle = (v_\ell * \widetilde{u}_\ell)(2k).$$

Therefore $\{R_{2^\ell k} f_\ell\}_{k=0}^{(N/2^\ell-1}$ and $\{R_{2^\ell k} g_\ell\}_{k=0}^{(N/2^\ell)-1}$ are orthonormal sets and are mutually orthogonal. Hence, the set (3.21) is orthonormal. □

Lemma 3.8 shows that a subspace generated by translates of a vector can be decomposed into two orthogonal subspaces, each generated by translates of new vectors. This result generalizes Theorem 3.8, where the original subspace is the whole space $\ell^2(\mathbb{Z}_N)$.

The importance of Lemma 3.8 is that this decomposition can be applied repeatedly. By iterating the splitting process, we obtain the multilevel structure underlying wavelet bases. To describe this construction, the following terminology is convenient.

Definition 3.9. *Suppose X is an inner product space and U and V are subspaces of X. If $U \perp V$, that is, $\langle u, v\rangle = 0$ for all $u \in U$ and $v \in V$, define*

$$U \oplus V = \{u + v : \ u \in U, \ v \in V\}. \tag{3.24}$$

The space $U \oplus V$ is called the orthogonal direct sum *of U and V. In particular, the statement $U \oplus V = X$ means that U and V are subspaces of X, $U \perp V$, and every $x \in X$ can be written in the form $x = u + v$ for some $u \in U$ and $v \in V$.*

Lemma 3.10. *Suppose N is divisible by 2^ℓ, $g_{\ell-1} \in \ell^2(\mathbb{Z}_N)$, and the set*

$$\{R_{2^{\ell-1}k} g_{\ell-1}\}_{k=0}^{(N/2^{\ell-1})-1}$$

is orthonormal and has $N/2^{\ell-1}$ elements. Suppose $u_\ell, v_\ell \in \ell^2(\mathbb{Z}_{N/2^{\ell-1}})$ and the system matrix $A_\ell(n)$ in equation(3.1) is unitary for $n = 0, 1, \ldots, (N/2^\ell)-1$. Define

$$f_\ell = g_{\ell-1} * U^{\ell-1}(v_\ell), \qquad g_\ell = g_{\ell-1} * U^{\ell-1}(u_\ell).$$

Define the spaces

$$V_{-\ell+1} = \text{span}\{R_{2^{\ell-1}k}g_{\ell-1}\}_{k=0}^{(N/2^{\ell-1})-1}, \tag{3.25}$$

$$W_{-\ell} = \text{span}\{R_{2^{\ell}k}f_{\ell}\}_{k=0}^{(N/2^{\ell})-1}, \tag{3.26}$$

and

$$V_{-\ell} = \text{span}\{R_{2^{\ell}k}g_{\ell}\}_{k=0}^{(N/2^{\ell})-1}. \tag{3.27}$$

Then

$$V_{-\ell} \oplus W_{-\ell} = V_{-\ell+1}. \tag{3.28}$$

Proof. By Lemma 3.8, the sets $\{R_{2^{\ell}k}g_{\ell}\}_{k=0}^{N/2^{\ell}-1}$ and $\{R_{2^{\ell}k}f_{\ell}\}_{k=0}^{N/2^{\ell}-1}$ are orthogonal. Hence $V_{-\ell} \perp W_{-\ell}$. Using the definitions of f_ℓ and g_ℓ, one obtains

$$R_{2^{\ell}k}g_{\ell} = \sum_{j=0}^{N/2^{\ell-1}-1} u_\ell(j)\, R_{2^{\ell-1}(j+2k)}g_{\ell-1}, \tag{3.29}$$

and

$$R_{2^{\ell}k}f_{\ell} = \sum_{j=0}^{N/2^{\ell-1}-1} v_\ell(j)\, R_{2^{\ell-1}(j+2k)}g_{\ell-1}. \tag{3.30}$$

Therefore every basis vector of $V_{-\ell}$ and $W_{-\ell}$ is a linear combination of basis vectors of $V_{-\ell+1}$. Hence $V_{-\ell} \subseteq V_{-\ell+1}$, $\quad W_{-\ell} \subseteq V_{-\ell+1}$. Since both $V_{-\ell}$ and $W_{-\ell}$ have dimension $N/2^{\ell}$, the orthogonal direct sum $V_{-\ell} \oplus W_{-\ell}$ has dimension $N/2^{\ell-1}$, which is the dimension of $V_{-\ell+1}$. Consequently, $V_{-\ell} \oplus W_{-\ell} = V_{-\ell+1}$. □

The spaces $V_{-\ell}$ are indexed by negative integers so that $V_{-\ell} \subseteq V_{-\ell+1}$, which reflects the nested structure of the wavelet decomposition. This notation is also consistent with the terminology used later for wavelets on $\mathbb{R}$.

Lemma 3.10 provides the key step in proving that the output of the analysis phase of a p-stage wavelet filter bank yields the coefficients of the input signal with respect to a p-stage wavelet basis.

Theorem 3.11. *Suppose N is divisible by 2^p, and $u_1, v_1, u_2, v_2, \ldots, u_p, v_p$ is a p-stage wavelet filter sequence. Define $f_1, f_2, \ldots, f_p, g_1, g_2, \ldots, g_p$ as in Definition 3.4. Then $f_1, f_2, \ldots, f_p, g_p$ generate a p-stage wavelet basis for $\ell^2(\mathbb{Z}_N)$.*

Proof. It is enough to prove that the set in Definition 3.7 is orthonormal, since it has N elements. Since $f_1 = v_1$ and $g_1 = u_1$, Theorem 2.9 implies that $\{R_{2k}f_1\}_{k=0}^{N/2-1} \cup \{R_{2k}g_1\}_{k=0}^{N/2-1}$ is orthonormal.

By induction and Lemma 3.8, the set $\{R_{2^{\ell}k}f_{\ell}\}_{k=0}^{N/2^{\ell}-1}$ is orthonormal for each $\ell = 1, 2, \ldots, p$, and $\{R_{2^{p}k}g_{p}\}_{k=0}^{N/2^{p}-1}$ is also orthonormal. It remains to prove orthogonality between elements belonging to different subsets. Let $m < \ell$. By

Lemma 3.10,

$$R_{2^\ell k} f_\ell \in W_{-\ell} \subseteq V_{-\ell+1} \subseteq \cdots \subseteq V_{-m},$$

and $R_{2^m j} f_m \in W_{-m}$. Since $V_{-m} \perp W_{-m}$, it follows that $R_{2^\ell k} f_\ell \perp R_{2^m j} f_m$. Similarly, for any $\ell \le p$, $R_{2^p k} g_p \in V_{-p} \subseteq V_{-\ell}$ and $R_{2^\ell j} f_\ell \in W_{-\ell}$. Again, Lemma 3.10 gives $V_{-\ell} \perp W_{-\ell}$, so these vectors are orthogonal. Thus the set in Definition 3.7 is an orthonormal set.. Since the set has N elements, it is an orthonormal basis for $\ell^2(\mathbb{Z}_N)$. □

Figure 3.6 illustrates the subspace decomposition described in Lemma 3.10. Starting with $\ell^2(\mathbb{Z}_N)$, the space is decomposed into orthogonal subspaces $V_{-1} \oplus W_{-1}$. The subspace V_{-1} is then further decomposed into $V_{-2} \oplus W_{-2}$, and the process continues recursively. At each stage the detail space $W_{-\ell}$ is retained, while the approximation space $V_{-\ell}$ is further decomposed. At the final stage, both V_{-p} and W_{-p} are retained.

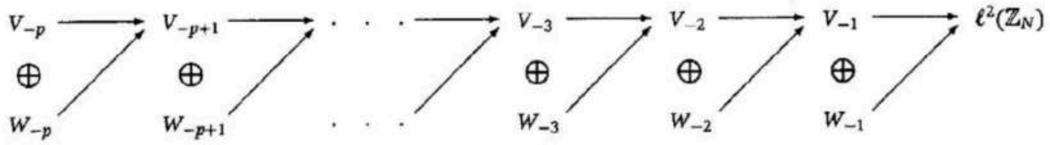

Figure 3.6:

By equations(3.18) and (3.19), the output of the analysis phase of the p-stage wavelet filter bank consists of the coefficients of the input signal with respect to the p-stage wavelet basis constructed in Theorem 3.11. Consequently, wavelet coefficients can be computed efficiently using the filter bank algorithm. For comparison with later developments, we introduce the following notation.

Definition 3.12. *Suppose N is divisible by 2^p. Let $u_1, v_1, \ldots, u_p, v_p$ be a p-stage wavelet filter sequence, and let $f_1, g_1, \ldots, f_p, g_p$ be defined as in Definition 3.4. For $j = 1, 2, \ldots, p$ and $k = 0, 1, \ldots, (N/2^j) - 1$,*

define

$$\psi_{-j,k} = R_{2^j k} f_j, \tag{3.31}$$

and

$$\phi_{-j,k} = R_{2^j k} g_j. \tag{3.32}$$

Thus the p-stage wavelet basis generated by $f_1, f_2, \ldots, f_p, g_p$ has the form

$$\{\psi_{-1,k}\}_{k=0}^{(N/2)-1} \cup \{\psi_{-2,k}\}_{k=0}^{(N/4)-1} \cup \cdots \cup \{\psi_{-p,k}\}_{k=0}^{(N/2^p)-1} \cup \{\phi_{-p,k}\}_{k=0}^{(N/2^p)-1}. \tag{3.33}$$

The elements of this orthonormal basis are called wavelets on $\mathbb{Z}_N$

$$V_{-j} = \operatorname{span}\{\phi_{-j,k}\}_{k=0}^{N/2^j-1}, \tag{3.34}$$

$$W_{-j} = \operatorname{span}\{\psi_{-j,k}\}_{k=0}^{N/2^j-1}, \tag{3.35}$$

The vectors $\psi_{-j,k}$ and $\phi_{-j,k}$ are called wavelets on $\mathbb{Z}_N$.

Recipe 3.13

Suppose $2^p \mid N$. Let $u_1, v_1, \ldots, u_p, v_p$ be a p-stage wavelet filter sequence (Definition 3.1). Define $f_1, f_2, \ldots, f_p, g_1, g_2, \ldots, g_p$ as in Definition 3.4, and define $\psi_{-j,k}$ and $\phi_{-j,k}$ by equations (3.31) and (3.32). Then the set

$$\{\psi_{-1,k}\}_{k=0}^{N/2-1} \cup \{\psi_{-2,k}\}_{k=0}^{N/4-1} \cup \cdots \cup \{\psi_{-p,k}\}_{k=0}^{N/2^p-1} \cup \{\phi_{-p,k}\}_{k=0}^{N/2^p-1}$$

is a p-stage wavelet basis for $\ell^2(\mathbb{Z}_N)$.

Recipe 3.13 shows that a p-stage wavelet filter sequence generates a wavelet basis for $\ell^2(\mathbb{Z}_N)$. The corresponding filter bank decomposition provides an efficient method for computing the wavelet coefficients of a signal and for reconstructing the original signal from these coefficients. The ability of wavelet bases to represent signals efficiently makes them useful in a variety of applications. In the next chapter, we discuss their application to image compression.

Chapter 4

WAVELETS AND IMAGE COMPRESSION

4.1 Image Compression

One of the most important applications of wavelets is image compression. In digital imaging, a picture is represented as a collection of small squares called pixels. Each pixel is assigned a numerical value corresponding to its brightness or gray-scale level. Thus, an image can be viewed as a large matrix of numbers or, equivalently, as a vector in a high-dimensional space.

For example, an image consisting of 200×250 pixels contains 50,000 pixel values. Storing and transmitting such a large amount of data requires significant memory and bandwidth. Therefore, efficient compression techniques are essential.

Wavelet bases provide an effective method for image compression. When an image is expanded with respect to a wavelet basis, much of the important information is concentrated in a relatively small number of coefficients, while many coefficients are very small. These small coefficients can be discarded with little loss of visual quality. The image can then be reconstructed from the remaining coefficients, resulting in a compressed representation.

Compared with Fourier methods, wavelets possess both spatial and frequency localization. This allows them to represent edges and local features of images more efficiently, making wavelets a powerful tool in modern image compression techniques.

4.2 Wavelets versus Fourier Compression

Wavelet-based image compression often produces better visual results than Fourier-based compression when the same number of coefficients is retained.

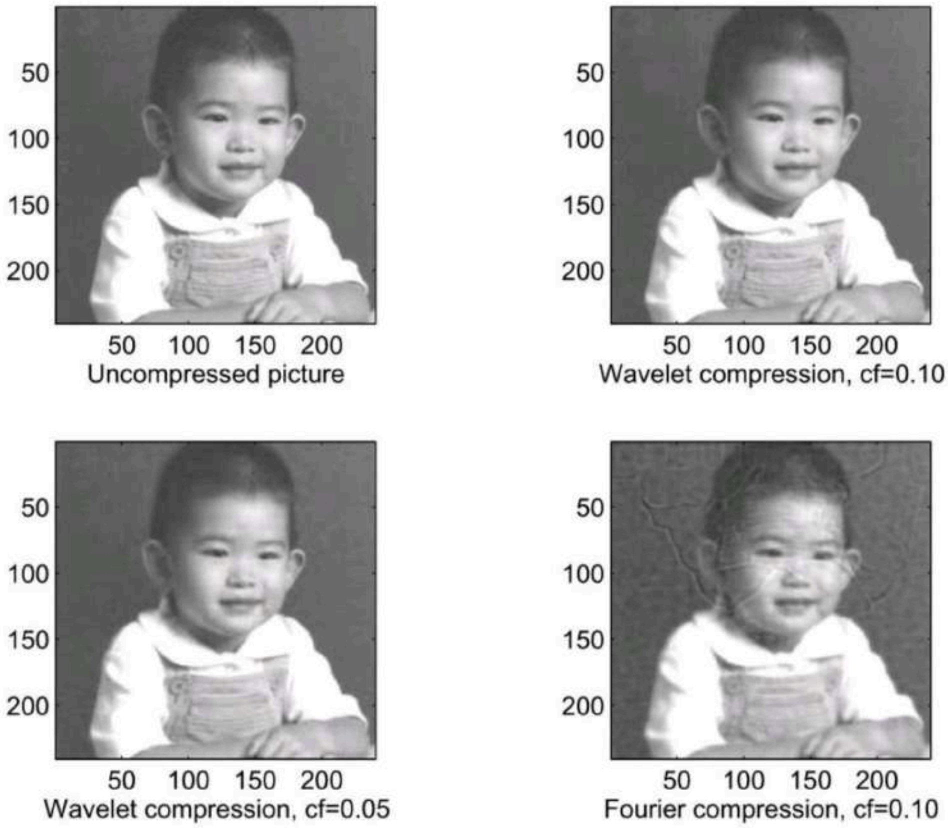

Figure 4.1: Comparison of wavelet and Fourier image compression.

Figure 4.1 compares image compression using wavelet and Fourier bases. When only a small percentage of the largest wavelet coefficients are retained, important image features such as edges and boundaries remain relatively sharp. In contrast, Fourier compression may introduce oscillatory artifacts near sharp transitions, resulting in a blurrier reconstruction.

The main reason for this difference is localization. Fourier basis functions extend over the entire image and therefore do not represent local features efficiently. Wavelet basis functions, on the other hand, are localized in both space and frequency, allowing them to capture edges and fine details with fewer coefficients. Consequently, wavelet compression generally preserves local image features more effectively than Fourier compression.

The upper-left image is the original uncompressed picture. The upper-right image is obtained by retaining the largest 10% of the wavelet coefficients, while the lower-left image retains only 5% of the wavelet coefficients. The lower-right image shows the result of retaining the largest 10% of the Fourier coefficients.

The wavelet-compressed images preserve important local features, such as edges and boundaries, even when only a small number of coefficients is retained. In contrast, the Fourier-compressed image appears blurrier and exhibits oscillatory artifacts near sharp transitions. This occurs because Fourier basis functions are spread over the entire image and are not spatially localized.

Wavelets possess both spatial and frequency localization. As a result, local image features affect only a small number of wavelet coefficients, allowing the image to be represented efficiently using fewer coefficients. This makes wavelets particularly effective for image compression and signal representation.

4.3 Wavelets and Image Compression

Wavelets have become an important tool in image compression because they provide both spatial and frequency localization. Unlike Fourier-based methods, which represent an image using basis functions spread over the entire image, wavelets represent local features such as edges and textures more efficiently.

Traditional JPEG compression divides an image into small blocks and applies a discrete cosine transform to each block. At high compression ratios, this may produce visible block artifacts near the boundaries of the blocks. Wavelet-based compression avoids this problem by using a multiscale representation of the image, resulting in smoother reconstructions and better preservation of image details.

Experimental comparisons show that wavelet compression often produces higher visual quality than JPEG compression at large compression ratios. Consequently, wavelets have become an important tool in modern image processing and data compression.

4.4 Comparison with JPEG Compression

Figure 4.2 compares JPEG and wavelet-based image compression. Figure 36(a) shows the original image. Figures 36(b) and 36(c) show JPEG compression at ratios of approximately $10:1$ and $40:1$, respectively, while Figure 36(d) shows JPEG compression at a ratio of about $103:1$. Figures 36(e) and 36(f) show wavelet-based compression at compression ratios comparable to those of Figures 36(c) and 36(d).

At moderate compression ratios, both JPEG and wavelet methods produce acceptable image quality. However, as the compression ratio increases, JPEG compression introduces noticeable visual distortions and blocking artifacts. In contrast, the wavelet-based method preserves image details more effectively and produces a smoother reconstruction.

The superior performance of wavelet compression is a consequence of the multiscale structure of wavelet bases. Since wavelets are localized in both space and frequency, important image features can be represented using relatively few coefficients. As a result, wavelet methods generally provide better image quality at high compression ratios than traditional JPEG compression.

Figure 4.2: Comparison of JPEG and wavelet-based image compression. (a) Original image. (b) JPEG compression at approximately 10:1. (c) JPEG compression at approximately 40:1. (d) JPEG compression at approximately 103:1. (e) Wavelet compression at a ratio comparable to (c). (f) Wavelet compression at a ratio comparable to (d).

Figure 4.2 illustrates that wavelet-based compression generally produces better visual quality than JPEG compression at high compression ratios. This advantage arises from the multiscale and localized nature of wavelet representations, which allow important image features to be preserved using relatively few coefficients.

Besides image compression, wavelets have numerous applications in science and engineering. They are used in signal processing, audio compression, medical imaging, fingerprint recognition, seismic data analysis, and wireless communications. Recent advances in artificial intelligence and machine learning have also created new applications for wavelet-based techniques in image analysis, feature extraction, and pattern recognition. These diverse applications demonstrate the effectiveness of wavelets as a powerful tool for the analysis, representation, and processing of complex data.

Conclusion

Wavelets on $\mathbb{Z}_N$ provide a powerful framework for the analysis and representation of discrete signals. By combining ideas from linear algebra, discrete Fourier analysis, and filter bank theory, it is possible to construct orthonormal bases that capture both frequency and spatial information efficiently.

The iterative filter bank approach leads naturally to multistage wavelet filter sequences and multiscale signal representations. Through the decomposition of orthogonal subspaces, these filter sequences generate orthonormal wavelet bases for $\ell^2(\mathbb{Z}_N)$, providing an effective means of representing discrete data.

The practical significance of wavelets was illustrated through image compression. Comparisons with Fourier and JPEG-based compression methods show that wavelets preserve important image features more effectively due to their localization properties. Consequently, wavelets provide both efficient computation and high-quality signal representation.

Thus, wavelets on $\mathbb{Z}_N$ combine a strong mathematical foundation with important practical applications, making them a valuable tool in modern signal and image processing.

Bibliography

[1] Michael W. Frazier, *An Introduction to Wavelets Through Linear Algebra*, Springer, New York, 1999.

[2] M. Thamban Nair and Arindama Singh, *Linear Algebra*, Springer, Singapore, 2018.

[3] Gilbert Strang and Truong Nguyen, *Wavelets and Filter Banks*, Wellesley–Cambridge Press, Wellesley, 1996.

[4] Ingrid Daubechies, *Ten Lectures on Wavelets*, SIAM, Philadelphia, 1992.