A Study on Geodesic Convexity and Related Parameters in Graphs

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A Study on Geodesic Convexity and Related Parameters in Graphs

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This is to certify that the dissertation entitled **"A STUDY ON GEODESIC CONVEXITY AND RELATED PARAMETERS IN GRAPHS"** is a record of studies carried out by **AALIYA A**, M.Sc Mathematics student of the year **2024-2026** at **T.K.M College of Arts and Science**, under my guidance and submitted to the University of Kerala in partial filfillment of the Degree of Master of Science in Mathematics.

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Introduction

Graph theory is a fundamental branch of discrete mathematics that studies the relationships among objects represented as vertices and edges. Since its inception, graph theory has developed into a powerful mathematical framework with extensive applications in computer science, communication networks, operations research, engineering, biology, social sciences, transportation systems, and many other disciplines. Owing to its ability to model complex systems and interactions, graph theory has become an indispensable tool for both theoretical investigations and practical problem solving.

One of the important concepts associated with graph theory is convexity, which originates from classical geometry. In Euclidean spaces, a set is said to be convex if the line segment joining any two points of the set lies entirely within the set. Convexity plays a central role in mathematical optimization, including linear programming, integer programming, nonlinear optimization, and optimal control. The significance of convexity lies in the fact that it restricts the search space for optimal solutions, making many optimization problems more tractable.

The axiomatic development of convexity during the twentieth century enabled the extension of geometric convexity to more abstract mathematical structures. As a result, several notions of convexity have been introduced in graph theory by replacing line segments with suitable classes of paths. These graph convexities provide a framework for studying structural properties of graphs and have found applications in network analysis, facility location, information dissemination, image processing, and communication networks.

Among the various graph convexities, geodesic convexity is one of the most widely studied and fundamental. In geodesic convexity, the role of a line segment is played by a shortest path (or geodesic) between two vertices. A subset of vertices is said to be geodesically convex if every shortest path between any two vertices of the subset is entirely contained within the subset. This notion naturally extends geometric convexity to graphs and provides a rich framework for investigating graph structure and optimization problems.

The study of geodesic convexity has led to the introduction of several convexity invariants, which quantify different aspects of the convex structure of graphs. These invariants not only characterize structural properties of graphs but also reveal important relationships among graph classes and optimization problems. Some of the most significant convexity invariants include the interval number, which measures the size of intervals generated by vertex pairs; the convexity number, which represents the cardinality of a largest proper convex set; the hull number, which is the minimum size of a set whose convex hull equals the entire vertex set; and the Helly number, which reflects the intersection properties of families of convex sets. These parameters have been extensively studied because of their theoretical significance as well as their applications in graph algorithms and network analysis.

The primary objective of this dissertation is to investigate geodesic convexity in graphs, with particular emphasis on these convexity invariants. The thesis explores their fundamental properties, interrelationships, computational aspects, and behavior within various classes of graphs. Through this study, the work aims to contribute to a deeper understanding of geodesic convexity and its role in the broader theory of graph convexity.

Background

Graph Theory

While studying the problem of Köinsberg bridge in 1735, Euler constructed a graph structure called Eulerian Graph. This is still considered as the origin of Graph Theory. In 1840, the complete graph and bipartite graphs were added to the literature of Graph Theory by A.F Möbius. By using complete graph and bipartite graphs as a tool to study recreational problems, Kuratowski inroduced the concept of the tree. Also he implemented the ideas of Graph Theory in studying currents in electrical networks. The famous four color problem was posed by Thomas Gutherie, in 1852. Then in 1856, by studying the cycles on polyhydra, Thomas. P. Kirkman and William R.Hamilton introduced the concept called Hamiltonian graph. In 1913, H.Dudeney inroduced several puzzle problems. After a century, in 1976 Kenneth Appel and Wolfgang Haken solved the four colour problem. This is said to be the birth of Graph Theory. The term graph in its present sense was used for the first time by Sylvester and it was derived in the context of graphic notation of chemical formulae. The usefulness of graphs in modeling network structures coupled with the advancement of computing techniques further enhanced the growth of graph theory from being a topic of study under combinatorics into a major area of study and research in mathematics. Later graph theory assumes as a unique position under applied mathematics in various fields like social networks, communication networks, traffic flow.

Convexity

The concept of convexity has been an area of interest in geometry since the time of Euclid. The notion of convex sets in d-dimensional Euclidean space, $\mathbb{R}^d$, is a classical concept that has a profound theory and various applications. J. Eckhoff developed the axiomatic theory of convexity, which extended the characteristics of convex sets in $\mathbb{R}^d$.Further theory on axiomatic approach was due to D.C. Kay and E.W.Womble. The axiomatic approach to convexity unified the concepts of convexity that exist in different mathematical structures, such as metric spaces, graphs, posets and matroids, by means of algebraic, combinatorial and geometric methods. Graph convexity is an area of mathematics that is concerned with studying the convexity properties of graphs, which are mathematical structures that model relationships between objects. A graph convexity is an abstract convexity defined on the vertex set of a connected graph.

The generalization of classical results about convex sets in $\mathbb{R}^n$ to abstract convexity spaces, defined by sets of paths in graphs, leads to many challenging structural and algorithmic problems. The most prominent types of graph convexities are defined in terms of paths in the graph. Geodesic convexity is also known as metric convexity or d-convexity. The convexity associated with the geodesic interval function I is called geodesic convexity, where $I(u, v)$ of a graph G is the set consisting of vertices lying on all geodesics (shortest paths) between u and v. The interval function I and the associated geodesic convexity are fundamental tools in metric graph theory

Objectives

The objective of this dissertation is to provide a comprehensive study of a particular type of graph convexity, Geodesic Convexity:

- To study geodesic convexity in graphs and understand its basic properties.
- To analyze the geodesic number, hull number, and convexity number for graphs.
- To examine the Carathèodory, Helly, and Radon numbers in the context of graph convexity.

The hull number of a graph was introduced by M. G. Everett and S. B. Seidman in the context of the well-known geodesic convexity, but only in the last decade the computational aspects of computing this parameter has gained interest.

Although these fields are well advanced, convexity studies are insubstantial in certain areas, and we attempt to fortify this with our thesis. The research expounded in this thesis is of great importance in numerous fields. It is hoped that this work will contribute to the development of new algorithms and approaches for solving challenging problems. May our work kindle the flames of inspiration among scholars and fuel future inquiry into the multifarious dimensions of graph convexity

Chapter 1

Preliminaries

This chapter presents the basic concepts, definitions, and notation from graph theory and convexity spaces, that will be used throughout this dissertation. Unless otherwise stated, all graphs considered in this dissertation are finite, undirected, connected, and simple; that is, they contain neither loops nor multiple edges.

1.1 Graph Theory

Definition 1.1.1. A **graph** is a finite non-empty set of objects called **vertices** together with a set of unordered pairs of distinct vertices of G, called **edges**. The vertex set and the edge set of G are respectively denoted by $V(G)$ and $E(G)$. A graph G with vertex set $V(G)$ and edge set $E(G)$ is denoted by $G = (V, E)$.
If $e = \{u, v\}$ is an edge, we write $e = uv$ and we say e joins the vertices u and v; u and v are **adjacent** vertices; u and v are incident with e. If two vertices are not joined, then we say that they are **non-adjacent**. If two distinct edges are incident with a common vertex, then they are said to be adjacent to each other.

Definition 1.1.2. The number of elements in the vertex set of a graph is called the **order** of G and is denoted by $|V|$. The number of elements in the edge set of a graph is called the **size** of G and is denoted by $|E|$

Definition 1.1.3. The **open neighborhood** of any vertex $v \in V(G)$ is the set of neighbors of v in G. Denoted by $N(v)$. i.e. ,$N(u) = \{v \in V(G) : uv \in E(G)\}$. The set $N[v] = N(v) \cup \{v\}$ is the **closed neighborhood** of v.

Definition 1.1.4. The **degree** of vertex $v \in V(G)$,denoted by $deg(v)$, is the number of neighbors of v. The **maximum degree** of G, denoted by $\Delta(G)$, is the largest degree among all vertices of G. The **minimum degree** of G, denoted by $\delta(G)$, is the smallest degree among all vertices of G. Vertices of degree 0 and 1 are called isolated vertices and end vertices, respectively.

Definition 1.1.5. A graph G called **regular** of degree r, or simply **r-regular** if $\delta(G) = \Delta(G)$. Let $[n]$ denote the set $\{1, ..., n\}$, for any positive integer n. The **complete graph** K_n is the graph of order n in which any two distinct vertices are adjacent. A **clique** of a graph is a maximal complete subgraph.

Definition 1.1.6. A **walk** of a graph G is an alternating sequence of vertices and edges, beginning and ending with vertices immediately preceding and following it. The walk $v_0e_1v_1e_2v_2 \cdots e_nv_n$ is a **closed walk** if $v_0 = v_n$, otherwise it is an **open walk**.

Definition 1.1.7. A graph H is called a **subgraph** of a graph G if $V(H) \subset V(G)$ and $E(H) \subset E(G)$. A **spanning subgraph** of G is a subgraph H with $V(H) = V(G)$. For any two vertices of any vertex set S of G, the **induced subgraph** H is the maximal subgraph of G with vertex set S. Therefore two vertices of S are adjacent in H if and only if they are adjacent in G. H is subgraph induced by S in G, we write $H = G[S]$.

Definition 1.1.8. A graph G_1 is **isomorphic** to a graph G_2 if there is a bijection f from $V(G_1)$ to $V(G_2)$ such that $uv \in E(G_1)$ if and only if $f(u)f(v) \in E(G_2)$. If G_1 is isomorphic to G_2, we write $G_1 \cong G_2$.

Definition 1.1.9. The Complement $\overline{G}$ of a graph G is the graph with vertex set $V(G)$ such that two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in G.

Definition 1.1.10. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be any two graphs Then their **union** $G_1 + G_2$ is the graph whose vertex set is $V_1 \cup V_2$ and edge set is $E_1 \cup E_2$.

Definition 1.1.11. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be any two graphs Then their **join** $G_1 \cup G_2$ is the graph whose vertex set is $V_1 \cup V_2$ and edge set is $E_1 \cup E_2 \cup \{uv : u \in V_1, v \in V_2\}$.

Definition 1.1.12. A subset U of the vertex set in a graph G is said to be **independent** if no two vertices in U are adjacent.

Definition 1.1.13. A **bipartite** graph is a graph G whose vertex set $V(G)$ can be partitioned into two subsets U and W such that every edge in G has one end in U and the other end in W. The **complete bipartite** graph $K_{h,k}$ is the bipartite graph of order $n = h + k$ such that $V(K_{h,k}) = U \cup W$, $G[U] \cong \overline{K_h}$, $G[W] \cong \overline{K_k}$, and being every vertex in U adjacent to every vertex in W, i.e., $K_{h,k} = \overline{K_h} \vee \overline{K_k}$

Definition 1.1.14. A **connected** graph is a graph in which every two vertices are connected. A connected subgraph H of a graph G is a **component** of G if H is not a proper subgraph of any connected subgraph of G. A set S of vertices of a connected graph G is called a **cutset** of G if the graph $G - S$ is not connected. A vertex $v \in V(G)$ is a **cut-vertex** of G if $G - v$ is disconnected. The **connectivity** $\kappa(G)$ of a connected non-complete graph G is the minimum cardinality of a cutset of G.

1.2 Metric Graph Theory

Definition 1.2.1. For any two vertices $u, v \in V(G)$ of a connected graph G, a **u-v geodesic** is a shortest $u - v$ path, i.e., a $u - v$ path of minimum order. The distance $d_G(u, v)$ between vertices u and v is the length of a $u - v$ geodesic.

Definition 1.2.2. The distance function $d_G : G \times G \to N$ associated to a connected graph G satisfies, for every $u, v, w \in V(G)$, the following properties:

1. $d(u, v) \geq 0$, equality holding if and only if $u = v$.

2. $d(u, v) = d(v, u)$.

3. $d(u, v) \leq d(u, w) + d(w, v)$.

Hence, for every connected graph G, d_G is a **metric** on $V(G)$ and the pair $(V(G), d_G)$ is a **metric space**.

Definition 1.2.3. For any induced subgraph H of a graph G, which holds the equality $d_G(u, v) = d_H(u, v)$ for any $u, v \in V(H)$, then H is said to be an **isometric subgraph** of G. An injective mapping f from the vertex set $V(H)$ of a graph H to the vertex set $V(G)$ of a graph G is called an **isometric embedding** if $d_G(f(u), f(v)) = d_H(u, v)$ for any $u, v \in V(H)$.

Definition 1.2.4. The **eccentricity** $e(v)$ of a vertex $v \in V(G)$ is the maximum distance between v and any other vertex of G, i.e., $e(v) = max\{d(v, u) : u \in V(G)\}$. The **radius** $rad(G)$ of G is the smallest eccentricity among the vertices of G, i.e., $rad(G) = min\{e(v) : v \in V(G)\}$.

Definition 1.2.5. A **central vertex** is any vertex v such that $e(v) = rad(G)$. The **center** $Cen(G)$ is the subgraph induced by the set of central vertices of G.

Definition 1.2.6. The **diameter** $diam(G)$ of G is the maximum distance between two vertices of G, i.e., $diam(G) = max\{e(v) : v \in V(G)\}$.

Definition 1.2.7. Any vertex v such that $e(v) = diam(G)$ is called **peripheral**, whereas any two vertices at a distance equal to the diameter of the graph are said to be **antipodal**. A vertex v is peripheral if and only if the positive integer-valued function $e : V(G) \rightarrow N$ has a global maximum at v. The **periphery** $Per(G)$ is the subgraph induced by the set of peripheral vertices of G.

Definition 1.2.8. A vertex $v \in V(G)$ is called **locally peripheral** if e has a local maximum at v, i.e., if $e(v) \geq e(u)$ for every $u \in N(v)$. The **contour** $Ct(G)$ is the subgraph induced by the set of locally peripheral vertices of G.

Definition 1.2.9. A vertex $v \in V(G)$ is called **eccentric** if there exists another vertex $u \in V(G)$ such that no vertex in the entire graph is further away from u than v. The **eccentric set** $Ecc(u)$ of a vertex u is the set of all its eccentric vertices. The **eccentric subgraph** $Ecc(G)$ is the subgraph induced by the set of eccentric vertices of G.

Definition 1.2.10. A vertex $v \in V(G)$ is said to be **locally eccentric** if, for some vertex $u \in V(G)$ if there exists another vertex $u \in V(G)$ such that no neighbor of v is further away from u than v. By $\delta(u)$ we denote the set of all locally eccentric vertices of a given vertex u, and the vertices of $\delta(u)$ are said to be the **maximally distant** vertices from u.

Definition 1.2.11. The **boundary** $\delta(G)$ is the subgraph induced by the set of locally eccentric vertices of G.

Definition 1.2.12. A vertex v of a graph G is called **simplicial** if its neighborhood $N(v)$ induces a clique. The **extreme subgraph** $Ext(G)$ is the subgraph induced by the set of simplicial vertices of G

Proposition 1.1. Let $G = (V, E)$ be a connected graph.

1. If $|Per(G)| = |Ct(G)| = 2$, then either $|\delta(G)| = 2$ or $|\delta(G)| \geq 4$.

2. If $|Ecc(G)| = |Per(G)| + 1$, then $|\delta(G)| > |Ecc(G)|$.

3. If $|Ecc(G)| > |Per(G)|$, then $|\delta(G)| \geq |Per(G)| + 2$.

1.3 Convexity Space

Definition 1.3.1. A convexity $\mathcal{C}$ on a nonempty set V is a collection of subsets of V such that:

- $\emptyset, V \in \mathcal{C}$.

- Arbitrary intersections of convex sets are convex

- Every nested union of convex sets is convex

Definition 1.3.2. A **convexity space** is an ordered pair $(V, \mathcal{C})$, where V is a nonempty set and $\mathcal{C}$ is a convexity on V. The members of $\mathcal{C}$ are called **convex sets**.

Definition 1.3.3. a set $S \subset V$, the smallest convex set $[S]_{\mathcal{C}}$ containing S is called the **convex hull** of S. If $[S]_{\mathcal{C}} = V$, then S is said to be **$\mathcal{C}$-hull set** of V

Definition 1.3.4. A convexity space (V, C) and a convex set $W \subseteq V$, a vertex $v \in W$ is called an **extreme point** of W if the set $W \setminus \{v\}$ is also convex. Every convex set is the convex hull of its extreme points.

Theorem 1.1. *For every convexity space $(V, \mathcal{C})$, the following three statements are equivalent:*

- *$(V, \mathcal{C})$ has the anti-exchange property.*
- *If $W \in \mathcal{C}$ and $W \neq V$, then $W \cup \{y\} \in \mathcal{C}$ for some $y \in V - W$.*
- *$(V, \mathcal{C})$ is a convex geometry.*
- *If $W \in \mathcal{C}$ and $x \in W$, then x is an extreme point of $[W \cup \{x\}]_{\mathcal{C}}$.*

Definition 1.3.5. A graph convexity space is an ordered pair $(G, \mathcal{C})$, formed by a connected graph $G = (V, E)$ and a convexity $\mathcal{C}$ on V such that $(V, \mathcal{C})$ is a convexity space satisfying :

- $\emptyset, V \in \mathcal{C}$.
- Arbitrary intersections of convex sets are convex
- Every nested union of convex sets is convex
- Every member of $\mathcal{C}$ induces a connected subgraph of G

Definition 1.3.6. Let ϕ be a property to be checked on the set of paths in a graph G. A ϕ-path is a path having property ϕ. A path property is called **feasible** if, for every pair of distinct vertices $u, v \in V(G)$, there is a ϕ-path joining u and v.

The most natural graph convexities are the path convexities defined by a family $\mathcal{P}_\phi$ of paths, being ϕ a feasible path property.

Definition 1.3.7. Geodesic convexity is the path convexity obtained when $\mathcal{P}_\phi$ is the set of all geodesics, i.e., when ϕ is the property of being a shortest path. Also known as metric convexity and d-convexity.

Definition 1.3.8. For two vertices u and v of a graph G, a vertex $x \in V(G)$ is said to be **geodominated** by the pair $\{u, v\}$ if x lies on some $u - v$ geodesic in G. The **geodetic interval** $I_G[u, v]$ consists of u, v together with all vertices geodominated by the pair u, v.

Definition 1.3.9. If S is a set of vertices of G, then the **geodetic closure** $I_G[S]$ is the union of all sets $I[u, v]$ for $u, v \in S$

Chapter 2

Geodesic Convexity and Convexity Invariants

Convexity is a fundamental concept in mathematics that has been extensively studied in areas such as geometry, optimization, combinatorics, and computational mathematics. In Euclidean spaces, a set is said to be convex if, for every pair of points in the set, the line segment joining them is entirely contained within the set. This simple geometric idea has inspired numerous generalizations in discrete mathematics, leading to the development of several notions of convexity on graphs. In graph theory, the concept of convexity is obtained by replacing line segments with suitable classes of paths, giving rise to different graph convexities such as geodesic convexity, monophonic convexity, induced path convexity, and Steiner convexity. Each of these convexities captures different structural properties of graphs and has found applications in various areas of mathematics and computer science.

Among these, *geodesic convexity* is one of the most natural and extensively investigated graph convexities. It is based on the shortest-path metric of a graph, where shortest paths, or geodesics, play the role of line segments in Euclidean geometry. A subset of vertices is said to be geodesically convex if it contains every vertex lying on every shortest path between any two of its vertices. This notion provides a metric perspective for studying the structure of graphs and establishes a close connection between graph theory and classical convexity theory. Because shortest paths are fundamental to the analysis of networks, geodesic convexity has become an important tool in studying communication networks, transportation systems, social networks, biological networks, facility location problems, and information dissemination.

One of the principal objectives in the study of graph convexity is to understand how the convex structure of a graph can be generated and characterized. This has led to the introduction of several numerical parameters, commonly referred to as *convexity invariants*, which quantify different aspects of geodesic convexity. These invariants provide valuable information about the structural complexity of graphs and the behavior of their convex sets. They also reveal interesting relationships between graph convexity and classical graph-theoretic concepts.

Among the most important convexity invariants are the *hull number* and the *geodetic number*. The hull number of a graph is the minimum cardinality of a set whose convex hull is the entire vertex set, whereas the geodetic number is the minimum cardinality of a set such that every vertex of the graph lies on a shortest path joining two vertices of the set. Although these two parameters are closely related, they capture different aspects of the convex generation process and often exhibit significantly different behaviors on various graph classes. Another important parameter is the *convexity number*, which measures the maximum cardinality of a proper convex subset of a graph. The study of these parameters has led to numerous structural characterizations, extremal results, and computational problems.

In addition to these primary invariants, several parameters originating from classical convexity theory have natural analogues in graph convexity. The *Helly number* describes the intersection properties of convex sets, the *Radon number* measures the partition properties of vertex sets with respect to convex hulls, and the *Carathèodory number* concerns the minimum number of vertices required to generate points in a convex hull. These invariants establish deep connections between discrete convexity and classical convexity theory, enriching the study of geodesic convexity and providing powerful tools for analyzing graph structures.

The study of convexity invariants has attracted considerable attention because these parameters not only characterize the convex structure of graphs but also interact with many classical graph parameters, including diameter, radius, clique number, independence number, and the set of extreme vertices. Moreover, determining these invariants is computationally challenging for many graph classes, making them an active area of research in graph theory and combinatorial optimization.

In this chapter, we present the fundamental concepts of geodesic convexity and discuss the principal convexity invariants associated with it. We begin with the definitions and basic properties of geodesic intervals, convex sets, and convex hulls. We then introduce hull sets, geodetic sets, and the corresponding invariants, namely the hull number and the geodetic number, followed by a study of the convexity number. Finally, we briefly discuss the Helly number, Radon number, and Carathèodory number, together with their fundamental properties and relationships.

2.1 Geodetic and Hull Numbers

Definition 2.1.1. A set S of vertices is called **geodesically convex**, **g -convex**, or simply **convex**, if $I[S] = S$, i.e., if for every pair $u, v \in S$, the interval $I[u, v] \subseteq S$.

Definition 2.1.2. $I[S] = V(G)$, then S is said to be a **geodetic set**.

Definition 2.1.3. The **convex hull** $[S]$ of a set S of vertices is the smallest convex set containing S. It is also called the **geodesic convex hull** and denoted by $[S]_g$.

Proposition 2.1. [6] Let S be a non-convex set of a graph G. Then, $S \subset I[S] \subseteq [S] \subseteq V(G)$. Moreover, the following statements are equivalent:

- $[S]$ is the smallest convex set containing S.
- $[S]$ is the intersection of all convex sets containing S.
- $[S] = I^k[S]$, for every positive integer $k \geq gin(S)$.

Definition 2.1.4. If a set S satisfies $[S] = V(G)$, then it is called a **hull set** of G.

Definition 2.1.5. The **geodetic number** of a graph G, denoted by $g(G)$, is the minimum cardinality of a geodetic set of $V(G)$.

Definition 2.1.6. The **hull number** of a graph G, denoted by $h(G)$, is the minimum cardinality of a hull set of $V(G)$.

If $G = \sum_{i=1}^{p} G_i$ is a non-connected graph with p components, then $g(G) = \sum_{i=1}^{p} g(G_i)$ and $h(G) = \sum_{i=1}^{p} h(G_i)$.

Theorem 2.1. *[6] If G is a nontrivial graph of order n and diameter d, then $2 \leq h(G) \leq g(G) \leq n - d + 1$.*

Proof. Every nontrivial graph satisfies $2 \leq h(G) \leq g(G)$, since every geodetic set is a hull set and the unique graph for which $h(G) = 1$ is $G \cong K_1$. Let u, v be vertices of G for which $d(u, v) = d$ and ρ a $u - v$ geodesic. If $V(\rho) = \{u, v_1, ..., v_{d-1}, v\}$ and $S = V(G) \setminus \{v_1, ..., v_{d-1}\}$, then $I[S] = V(G)$ and, consequently, $g(G) \leq |S| = n - d + 1$. $\square$

Corollary 2.2. *[6] The only connected graph of order and geodetic number n is the complete graph K_n.*

Theorem 2.3. *[6] If n, d, and k are integers such that $2 \leq d < n$, $2 \leq k < n$, and $n - d - k + 1 \geq 0$, then there exists a graph G of order n, diameter d, and $h(G) = g(G) = k$.*

Proof. Let P_{d+1} a path of order $d + 1$ such that $V(P_{d+1}) = \{u_0, u_1, ..., u_d\}$. We first add $k - 2$ new vertices $v_1, v_2, ..., v_{k-2}$ to P_{d+1}, and join each to u_1, producing a tree T. Then, we add $n - d - k + 1$ new vertices $w_1, w_2, ..., w_{n-d-k+1}$ and join each to both u_0 and u_2, thereby producing the graph G of order n and diameter d displayed in Figure 2.1. Finally, $S = \{u_0, u_d, v_1, ..., v_{k-2}\}$ is both a minimum geodetic set and a minimum hull set of G $\square$

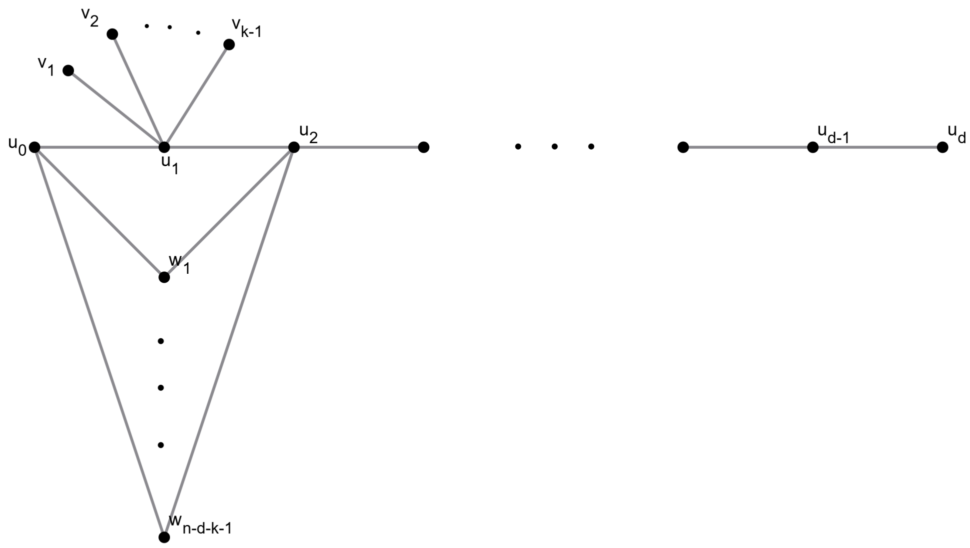

Figure 2.1: $h(G) = g(G) = k$.

Theorem 2.4. *[4] Let G be a connected graph with p points. If k is the number of extreme points of G, then*

$$k \leq h(G) \leq p - d(G) + 1.$$

Proof. Obviously every extreme point must be a member of each minimum hull set, and the lower bound follows. The upper bound follows from the fact that $h(G)$ must be smaller than the total number of points in G minus the number of points which are not endpoints on the longest geodesic. $\square$

Corollary 2.5. *[4] If G is a connected graph with p points, then $h(G) = p$ if and only if G is complete.*

Theorem 2.6. *[4] Let G be a connected graph. Then a vertex v is extreme if and only if the subgraph $\langle N(v) \rangle$ induced by $N(v)$ is complete.*

Proof. Suppose v is extreme and that $\langle N(v) \rangle$ is not complete. Then there exist points $x, y \in N(v)$ with $d(x, y) \geq 2$. The shortest path from x to y must be xvy, which contradicts the assumption that v is extreme.

Conversely, suppose that $\langle N(v) \rangle$ is complete and v is not extreme. Then $V - \{v\}$ is not convex,, so that there exist points $x, y \in N(v)$ with $d(x, y) = 2$, contradicting the completeness of $\langle N(v) \rangle$. $\square$

Theorem 2.7. *[4] Let G be a connected graph with p points, where $p \geq 3$. Then $h(G) = p - 1$ if and only if $d(G) = 2$, G contains only one cutpoint, and each block of G has diameter less than or equal to 1.*

Proof. We first suppose $d(G) = 2$, G contains only one cutpoint and each block of G has diameter 1. It follows from Theorem 2.4 that each point except the cutpoint is extreme, which easily implies the desired result.

Conversely, suppose that $h(G) = p - 1$. By Theorem 2.4, $d(G) \leq 2$, and hence Corollary 2.5 implies that $d(G) = 2$. Since $d(G) = 2$, there exists a geodesic of length 2, say uvw. If G has not cutpoints, u and w lie in the same block and hence on a common cycle. Since u and w are not adjacent, there exists a geodesic uxw with $x \neq v$. Consequently $\{u, v, w, x\} \subset (\{u, w\})$, so that $h(G) \leq p - 2$, a contradiction. Thus G cannot be a block. Since $d(G) = 2$, we must have exactly one cutpoint, which we will call c, and c must be adjacent to every point of G. This last remark means that c cannot lie in the minimum hull set. If the diameter of a block is equal to 2, then there exists a geodesic uxw with $x \neq c$. Hence x cannot be in a minimum hull set, once again implying that $h(G) \leq p-2$, a contradiction. □

Theorem 2.8. *[6] A vertex x is an extreme vertex with respect to the geodesic convexity in a graph G if and only if it is simplicial.*

Proof. Let x be an extreme vertex. If $deg_G(x) = 1$, then x is trivially simplicial. Suppose that $deg_G(x) \geq 2$ and take a pair of vertices $u, v \in N(x)$. Since $V - x$ is convex, every vertex lying on some $u - v$ geodesic must belong to $V - x$, which is only true if u and v are adjacent.

Conversely, assume that x is a simplicial vertex of G and take a pair of vertices $u, v \in V(G) - x$. Let ρ be a $u - v$ geodesic containing vertex x and take the neighbors $\{a, b\}$ of x belonging to $V(\rho)$. Notice that $d_G(a, b) = 2$, a contradiction since $N(x)$ induces a clique. □

Remark. Every graph G satisfies $|Ext(G)| \leq h(G) \leq g(G)$.

Definition 2.1.7. An **extreme geodesic graph** is a graph G such that $|Ext(G)| = g(G)$.

In these graphs, the set $Ext(G)$ of simplicial vertices is both its unique minimum geodetic and hull set. Complete graphs and trees provide two basic examples of extreme geodesic graphs.

Theorem 2.9. *[6] For every pair a,b of integers with $(a, b) \neq (0, 1)$ and $0 \leq a \leq b$, there exists a graph G such that $|Ext(G)| = a$ and $g(G) = b$.*

Proof. If $1 \leq a = b$, then Ka satisfies $|Ext(K_a)| = g(K_a) = a$. Thus we assume that $0 \leq a < b$ and $2 \leq b$. Take a copies $\{Fi\}_{i=1}^{a}$ of K_2 and $b - a$ copies $\{H_i\}_{i=1}^{b-a}$ of C_4. Let $V(F_i) = \{x_i, y_i\}$, $V(H_j) = \{u_j, v_j, w_j, z_j\}$, and $E(H_j) = \{u_j v_j, v_j w_j, w_j z_j, z_j u_j\}$ and consider the graph $G_{a,b}$ obtained from these two families by identifying the vertices $\{x_i\}_{i=1}^{a} \bigcup \{u_j\}_{j=1}^{b-a}$. Clearly, $Ext(G_{a,b}) = \{y_i\}_{i=1}^{a}$. It is also easy to check that the set $\{y_i\}_{i=1}^{a} \bigcup \{w_j\}_{j=1}^{b-a}$ is a minimum geodetic set of $G_{a,b}$. Thus, $|Ext(G_{a,b})| = a$ and $g(G_{a,b}) = b$. $\square$

Theorem 2.10. *[6] Let G be a connected graph of order $n \geq 3$. Then, the following three statements are equivalent:*

1. *$h(G) = n - 1$.*

2. *$g(G) = n - 1$.*

3. *$G \cong K_1 \vee (K_{n1} + \cdots + K_{nr})$ where $r \geq 2$ and $n_1 + \cdots + n_r = n - 1$.*

Proof. We shall show that (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1).
(1) $\Rightarrow$ (2) The only connected graph such that $g(G) = n$ is the complete graph K_n. Therefore, $g(G) = n - 1$, as $h(G) \leq g(G)$ and $h(K_n) = n$.
(2) $\Rightarrow$ (3) Let S be a minimum geodetic set of G, where $V(G) - S = v$. We claim that v is adjacent to every vertex in S. For every geodesic ρ joining any two nonadjacent vertices u, v of G, the set $[V(G) - V(\rho)] \cup \{u, v\}$ is a geodetic set of G. This means, first, that $diam(G) = 2$ and, second, that v is adjacent to every pair of nonadjacent vertices of S. Therefore, if u is a vertex of S nonadjacent to vertex v, then it must be adjacent to all other vertices of S. Since S is a geodetic set,v lies on some $s - t$ geodesic of length 2, where $s, t \in S$. Finally, since $us, ut \in E(G)$, it follows that $u \in S$, a contradiction. Hence, as claimed, v is adjacent to every vertex in S. To show that $G \cong K_1 \vee (K_{n1} + \cdots + K_{nr})$, it suffices to observe that for any triple $x, y, z \in S$, if $xy, yz \in E(G)$, then $x, z \in E(G)$, since otherwise $d(x, z) = 2$, and hence $y = v, a$ contradiction.
(3) $\Rightarrow$ (1) All vertices of G but the cut-vertex of G are extreme vertices. Therefore, $h(G) = n - 1$, since $|Ext(G)| \leq h(G)$ and the only connected graph such that $h(G) = n$ is K_n. $\square$

Corollary 2.11. *[1] Every nontrivial graph G of order n with geodetic number $g(G) = n - 1$ is an extreme geodesic graph.*

Proof. In general, for each integer $l > 2$, $G = (K_{n1} + \cdots + K_{nr}) \cup K_l$ is an extreme geodesic graph since the unique minimum geodetic set S of G is the set of its extreme vertices, that is, $S = V(K_n1) + V(K_n2) + \cdots + V(K_nr)$. So G is an extreme geodesic graph with $ex(G) = g(G) = n - l$. In particular, if $l = 2$, then the graph G in $G = (K_{n1} + \cdots + K_{nr}) \cup K_l$ has geodetic number $n - 2$. Since every graph of order n with geodetic number n or $n - 1$ is an extreme geodesic graph. □

Theorem 2.12. *[6] For every pair k, n of integers with $2 \leq k \leq n - 2$ there exists a graph G of order n and geodetic number $g(G) = k$ that is not an extreme geodesic graph.*

Proof. We consider two cases.

Case 1: $2 \leq k \leq n - 3$.

Take $G_1 = K_{2,n-k-1}$ and $G_2 = K_{1,k-1}$. Consider the graph G obtained from G_1 and G_2 by identifying v and z, where $V(G_1) = \{u, v\} \cup \{u_i\}_{i=1}^{n-k-1}$ and $V(G_2) = \{z\} \cup \{v_i\}_{i=1}^{k-1}$ (see Figure 2.2). Clearly,

$$Ext(G) = \{v_i\}_{i=1}^{k-1}$$

It is also easy to check that the set $\{u\} \cup \{v_i\}_{i=1}^{k-1}$ is the unique minimum geodetic set of G. Thus,

$$|Ext(G)| = k - 1 < k = g(G)$$

Case 2: $2 \leq k = n - 2$.

Take $G_1 = K_2$ and $G_2 = K_{n-2}$. Consider the graph G obtained from G_1 and G_2 by adding the edges v_2v_3 and v_1v_4, where $V(G_1) = \{v_1, v_2\}$ and $V(G_2) = \{v_i\}_{i=3}^{n}$ (see Figure 2.2). Clearly,

$$Ext(G) = \{v_i\}_{i=4}^{n}$$

It is also easy to check that the set $\{v_1, v_2\} \cup \{v_i\}_{i=4}^{n}$ is a minimum geodetic set of

G. Thus,

$$|Ext(G)| = n - 4 < n - 2 = g(G)$$

□

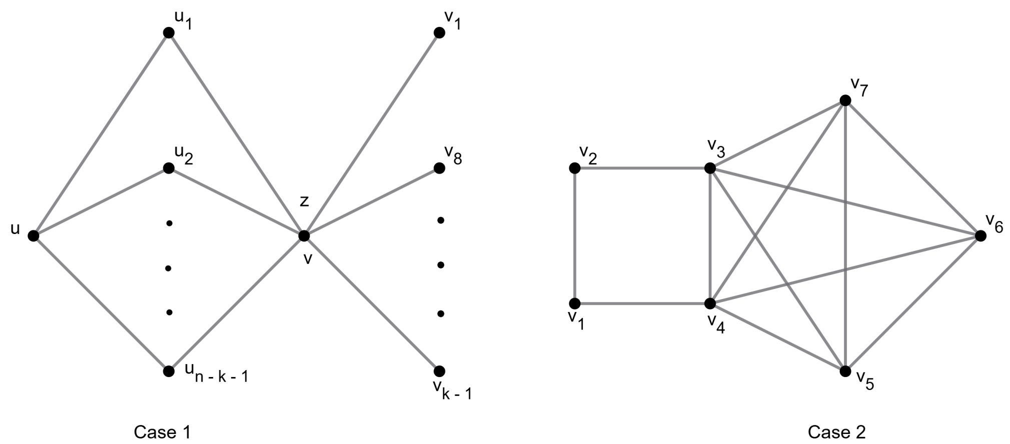

Figure 2.2: Case 1:$2 \leq k \leq n - 3$, Case 2: $2 \leq k = n - 2$

Theorem 2.13. *[6] If r, d, and k are integers such that $2 \leq k$ and $r \leq d \leq 2r$, then there exists an extreme geodesic graph G such that $rad(G)$ =r, $diam(G) = d$, and $h(G) = g(G) = k$.*

Proof. When $r = 1$, we let $G = K_k$ or $G = K_{1,k}$ according to whether $d = 1$ or $d = 2$, respectively. For $r \geq 2$, we construct an extreme geodesic graph G with the desired properties. Take the cycle C_{2r} and the path P_{d-r+1}, where$V(C_{2r}) = \{v_1, ..., v_{2r}\}$ and $V(P_{d-r+1}) = \{u_0, u_1, ..., u_{d-r}\}$. Let H be the graph obtained from C_{2r} and P_{d-r+1} by identifying $v_1 \in V(C_{2r})$ and $u_0 \in V(P_{d-r+1})$ and adding the edge $v_r v_{r+2}$. Finally, the graph G is then obtained by adding $k-2$ new vertices $w_1, w_2, ..., w_{k-2}$ to H and joining all of them to the vertex u_{d-r+1} (see Figure 2.3). Clearly, G is a graph of radius r and diameter d. It is also easy to check that $Ext(G) = \{v_{r+1}, w_1, ..., w_{k-2}, u_{d-r}\}$ is the sole geodetic set of G. Therefore, $g(G) = |Ext(G)| = k$, as desired. $\square$

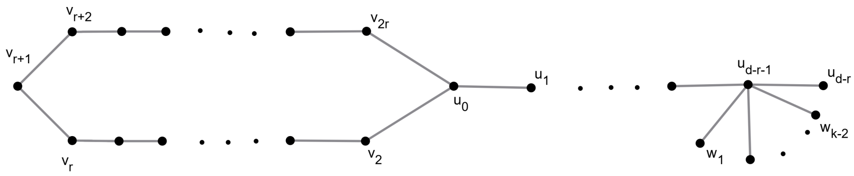

Figure 2.3: Extreme geodesic graph G s.t. $rad(G) = r$, $diam(G) = d$, and $h(G) = g(G) = k$

Definition 2.1.8. The **geodetic ratio** and **extreme ratio** of a graph G of order n are defined respectively as $r_g(G) = \frac{g(G)}{n}$ and $r_{ext}(G) = \frac{|Ext(G)|}{n}$.

Theorem 2.14. *[6] If n, d, and k are integers such that $2 \leq d < n$, $2 \leq k < n$, and $n - d - k + 1 \geq 0$, then there exists an extreme geodesic graph G of order n, diameter d, and $g(G) = k$.*

Proof. Take the graph $H \cong K_k \vee K_{n-d-k+2}$ where $V(K_k) = \{v_1, ..., v_k\}$ and $V(K_{n-d-k+2}) = \{w_1, ..., w_{n-d-k+2}\}$. Consider the path P_{d-1}, where $V(P_{d-1}) = \{u_0, ..., u_{d-2}\}$. Let G be the graph obtained from H and P_{d-1} by identifying $v_1 \in V(H)$ and $u_0 \in V(P_{d-1})$. Certainly, G is a graph of order n and diameter d. It is also easy to check that the set $Ext(G) = \{u_{d-2}, v_2, ..., v_k\}$ is the sole geodetic set of G. Therefore, $g(G) = |Ext(G)| = k$. $\square$

Theorem 2.15. *[6] For every pair s,t of rational numbers with $0 \leq s < t < \frac{1+s}{2} < 1$, there exists a connected graph G with $r_{ext}(G) = s$ and $r_g(G) = t$.*

Proof. First, we assume that $s > 0$. Let $s = \frac{s_1}{s_2}$ and $t = \frac{t_1}{t_2}$, where s_1, s_2, t_1, t_2 are positive integers. Since $0 < s < t < \frac{1+s}{2} < 1$, it follows that $s_2t_1 - s_1t_2 > 0$ and $s_2t_2 - 2s_2t_1 + s_1t_2 > 0$. Take an even integer $k > 1$ and consider the integers $a = ks_1t_2, k(s_2t_1 - s_1t_2)$, and $c = k(s_2t_2 - 2s_2t_1 + s_1t_2)$. Take $a - 1$ copies $\{F_i\}_{i=1}^{a}$ of K_2, b copies $\{H_i\}_{i=1}^{b}$ of $K_{2,3}$ and the path P_{c+1}. Consider the graph G obtained from all of these graphs by identifying the vertices $\{y_i\}_{i=1}^{a-1} \cup \{w_{j1}\}_{j=1}^{b} \cup \{v_{c+1}\}$, where

$$V(F_i) = \{x_i, y_i\},$$

$$V(H_j) = \{u_{j1}, u_{j2}\} \cup \{w_{j1}, w_{j2}, w_{j3}\}$$

and

$$V(P_{c+1}) = \{v_h\}_{h=1}^{c+1}$$

Clearly, the order of G is $n = a + 4b + c = ks_2t_2$. It is also easy to check that $Ext(G) = \{x_i\}_{i=1}^{a-1} \cup \{v_1\}$ and that the set $S = \{x_i\}_{i=1}^{a-1} \cup \{v_1\} \cup \{u_{j1}, u_{j2}\}_{j=1}^{b}$ is a minimum geodetic set of G. Hence,

$$|Ext(G)| = a = ks_1t_2$$

and

$$g(G) = a + 2b = ks_2t_1$$

i.e.,

$$r_{ext}(G) = s$$

and

$$r_g(G) = t$$

As for the case $s = 0$, it is similarly proved, by considering $b - 1$ copies of $K_{2,3}$ and a single copy of $K_{2,c}$, where $b = t_1$ and $c = 2t_2 - 4t_1 + 2$. □

2.2 Convexity Numbers

Definition 2.2.1. The **convexity number** of a connected graph G, denoted by $con(G)$, is the maximum cardinality of a proper convex set of G

Definition 2.2.2. The **clique number** $\omega(G)$ of a graph G is the maximum order of a clique in G.

Definition 2.2.3. The **independence number** $\alpha(G)$ is the cardinality of a largest independent set, i.e., $\alpha(G) = \omega(G)$.

Every non-complete graph G satisfies $2 \leq \omega(G) \leq con(G) \leq n - 1$, as the vertex set of every clique is convex. $con(G) = n - 1$ if and only if G contains at least a simplicial vertex v since $V(G) - v$ is a convex set if and only if $N(v)$ induces a clique.

Theorem 2.16. *[6] Let G be a graph of diameter $diam(G) = 2$. If its complement $\overline{G}$ is not connected and at least two of the components of $\overline{G}$ are nontrivial, then $con(G) = \omega(G)$.*

Proof. Let $\{H_i\}_{i=1}^{k}$ de components of $\overline{G}$. Certainly, $G = \vee_{i=1}^{k}\overline{H_i}$. Let C be a maximum proper convex set in G. Let $S_i = C \cap V(H_i)$, for every $i \in [k]$. To end the proof, it is enough to show that if $S_i \neq \emptyset$, then S_i induces a clique in $\overline{H_i}$. Suppose, to the contrary, that $S_1 \neq \emptyset$ and there exist two distinct vertices $u, v \in S_1$ such that $d_G(u, v) = 2$. Hence, $V(G) \setminus V(\overline{H_1}) \subseteq I_G[u, v] \subseteq C$. By hypothesis, there exists an index $j \neq 1$ such that H_j is nontrivial. Since H_i is connected, there exist two distinct vertices $a, b \in V(H_j)$ which are adjacent in H_j, i.e., a and b are not adjacent in $\overline{H_j}$. Hence, $d_G(a, b) = 2$ and both a and b are vertices in C adjacent to all vertices of $\overline{H_1}$, which means that $C = V(G)$, a contradiction. □

Definition 2.2.4. A graph G is called **polyconvex** if for every integer i with $1 \leq i \leq con(G)$, there exists a convex set of cardinality i in G.

Proposition 2.2. [6] For $n \leq 2$, a set S in the n-dimensional hypercube $\mathcal{Q}_n$ is convex if and only if S induces a k-dimensional hypercube $\mathcal{Q}_k$ in $\mathcal{Q}_n$, for some integer k with $0 \leq k \leq n$. Hence, for every $n > 2, \mathcal{Q}_n$ is not polyconvex.

Proof. If S is a set of vertices in $\mathcal{Q}_n$ that induces a $\mathcal{Q}_i$ for some $i \in [n]$, then S is convex in $\mathcal{Q}_n$. So it remains only to prove the converse. We proceed by induction on n. The result is certainly true for $n = 2$. Assume that every convex set in $\mathcal{Q}_k$ $(k \geq 2)$ induces a $\mathcal{Q}_i$ for some $i \in [k]$.

Let $\mathcal{Q}_{k+1}$ be a $(k+1)$-cube formed from two copies G_1 and G_2 of $\mathcal{Q}_k$, i.e., $\mathcal{Q}_{k+1} = \mathcal{Q}_k \Box K_2$, $V(G_1) = \mathcal{Q}_k \times \{1\}$ and $V(G_2) = \mathcal{Q}_k \times \{2\}$. Let S be a convex set in $\mathcal{Q}_{k+1}$. If either $S = V(\mathcal{Q}_{k+1})$ or $S \subseteq V(G_1)$ for $i = 1$ or $i = 2$, then the result holds. Thus we may assume that $S = S_1 \cup S_2$, where $S_1 = S \cap V(G_1)$ and $S_2 = S \cap V(G_2)$ are both nonempty. Certainly S_i is convex in G_i, for $i = 1, 2$. Hence, by induction hypothesis, $G[S_1] = \mathcal{Q}_r$ and $G[S_2] = \mathcal{Q}_s$, where $0 \leq r, s \leq k$. Take $u_1 \in S_1$ and $v_2 \in S_2$. If u_2 is the neighbor of u_1 in G_2 and v_1 is the neighbor of v_2 in G_1, then $u_2, v_1 \subseteq I[u_1, v_2]$.

Hence, $r = s$ and $S = \mathcal{Q}_{r+1}$. $\square$

Theorem 2.17. *[6] Given any pair of integers n, k with $2 \leq k \leq n - 1$, there exists a polyconvex connected K_3-free graph of order n with $con(G) = k$.*

Proof. For $k = 2$, take $G = K_{2,n-2}$. So we may assume that $3 \leq k \leq n - 1$. We consider three cases.

Case 1: $k + 1 \leq n \leq \frac{3}{2}k - \frac{1}{2}$. If $3 \leq k \leq 4$, P_4 and P_5. Suppose thus that $k \geq 5$. Take two copies G_1 and G_2 of $K_{2,n-k}$ and a copy G_3 of P_{2k-n-2}. Let G be the graph obtained from G_1, G_2, and G_3 by identifying the vertices w_1 and w_{2k-n-2} with y_1 and x_2, respectively, where $V(G_i) = \{x_i, y_i\} \cup \{z_{ij}\}_{j=1}^{n-k}$ and $V(G_3) = \{w_i\}_{i=1}^{2k-n-2}$ (in Figure 2.4a the case $n = 14$ and $k = 10$ is shown). Notice G is a K_3 - graph of order n, $2k - n - 2 \geq 0$, and if $n = k + 1$, then G is P_n. We form convex sets of orders 1 through $2k - n - 2$ by taking a subpath of appropriate length from G_3. We form convex sets of orders $2k - n - 1$ and $2k - n$ by taking all the vertices of G_3

and adding at most one vertex from each of the partite classes of order $n-k$ of G_1 and G_2. We form a convex set of orders $n-k+2$ to $k-1$ by taking all the vertices of G_1 together with an appropriate number of adjacent vertices of G_3. Finally, we form a convex set of order k by taking all the vertices of G_1 and G_3 and one more vertex from the partite class of order $n-k$ of G_2. Since $(n-k+2)-(2k-n) \leq 1$, we have found convex sets of all orders between 1 and $k = con(G)$.

Case 2: $\frac{3}{2}k \leq n \leq 2k-1$. Take a copy G_1 of $K_{2,2k-n}$ and a copy G_2 of P_{2n-2k}. Let G be the graph obtained from G_1 and G_2 by identifying the vertices w_1 and w_{2n-2k} with x and y, respectively, where $V(G_1) = \{x, y\} \cup \{z_i\}_{i=1}^{2k-n}$ and $V(G_2) = \{w_i\}_{i=1}^{2n-2k}$ (in Figure 2.4b the case $n = 12$ and $k = 8$ is shown). Notice (1) G is a K_3 - graph of order n, (2) $2k - n \geq 1$ and (3) if $n = 2k - 1$ then G is C_n. We form convex sets of orders 1 through $n - k + 1$ by taking a subpath of appropriate length from G_2. We form convex sets of orders $2 + 2k - n$ through k by taking all the vertices of G_1 and adding suitable numbers of adjacent vertices of G_2. Since $(2 + 2k - n) - (n - k + 1) \leq 1$, we have found convex sets of all orders between 1 and $k = con(G)$.

Case 3: $2k \leq n$. Let $m = \lfloor \frac{n-2k+4}{2} \rfloor$. Note that $m \geq 2$ since $n \geq 2k$. Start with a copy G_1 of $K_{m,m}$ or $K_{m,m+1}$, depending on whether $n - 2k + 4$ is even or odd. Let G be the graph obtained by replacing one edge of G_1 with a set Ω of $k - 2$ paths of length 3. Let v be one of the vertices of Ω of degree greater than 2. Let S be the set of vertices consisting of v, all the neighbors of v in Ω and one more neighbor of v. Then S is a (maximum) convex set of order k. Convex sets of all smaller orders occur as subsets of S containing v. Notice that when $k = 3$ and $n = 7$, then this construction does not work. For this case, take the graph $K_{3,3}$ and replace one edge with a path of length 2. Then the three vertices of this path form a largest convex set. $\square$

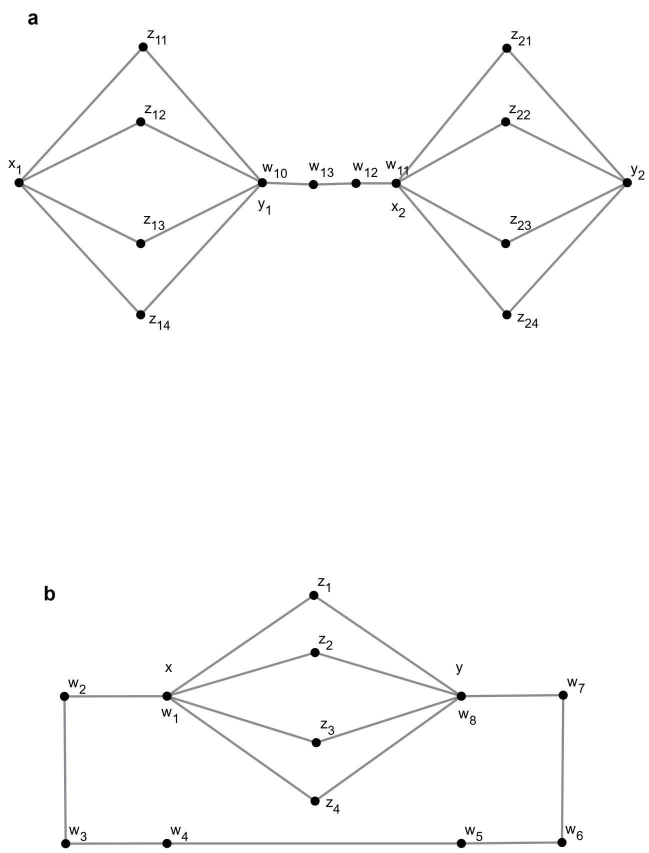

Figure 2.4: (a) Graph G s.t. $n = 14$ and $con(G) = k = 10$, (b) Graph G s.t. $n = 12$ and $con(G) = k = 8$

Chapter 3

Carathèdory and Helly Numbers

3.1 Carathèodory Number

Definition 3.1.1. A subset $A \subset V$ of a graph $G = (V, E)$ is called **redundant** when

$$[A] = \bigcup_{a \in A} [A - a]$$

and otherwise it is said to be **irredundant**.

Definition 3.1.2. The **Carathèodory number** $C(G)$ of a graph G is the maximum cardinality of an irredundant set.

Theorem 3.1. *[5] Every G_i, constructed as*

- *G_1 is the graph with $V(G_1) = \{v_1\}$, and $E(G_1) = \emptyset$;*
- *G_2 is the graph with*

$$V(G_2) = V(G_1) \cup \{v_2, v_3\}$$

and

$$E(G_2) = E(G_1) \cup \{v_1v_3, v_2v_3\}$$

- *G_3 is the graph with*

$$V(G_3) = V(G_2) \cup \{v_4, v_5, v_6\}$$

  and

$$E(G_3) = E(G_2) \cup \{v_1v_4, v_2v_4, v_3v_4, v_3v_5, v_4v_6, v_5v_6\}$$

- *G_{i+1}, for $i \geq 3$, is the graph with*

$$V(G_{i+1}) = V(G_i) \cup \{v_{n-2}, v_{n-1}, v_n\}$$

  and

$$E(G_{i+1}) = E(G_i) \cup \{v_{n-4}v_{n-2}, v_{n-3}v_{n-2}, v_{n-2}v_{n-1}, v_{n-1}v_n, v_4v_n\}$$

  where $n = 3i$.

has a Carathèodory set of cardinality i.

Proof. For $i = 1$, $S_1 = \{v_1\}$ is a Carathèodory set of G_1. For $i = 2$, $S_2 = \{v_1, v_2\}$ and $v_3 \in \delta H(S_2)$. We will now show by induction that

$$S_i = \{v_1, v_2\} \cup \{v_{3j} | 2 \leq j \leq i\}$$

is a Carathèodory set of G_i, with cardinality i, for $i \geq 3$. For $i = 3$, note that $v_5 \in \delta H(S_3)$. For $i = 4$, we have that $v_7, v_8 \in \delta H(S_4)$ and for $i = 5$, one can see that $\delta H(S_5) = \{v_{10}, v_{11}\}$. This proves that S_i is a Carathèodory set of G_i, with cardinality i, for $1 \leq i \leq 5$.

Take $i \geq 5$ and assume (as inductive hypothesis) that the longest geodesic of G_i is at most 4, and S_i is a Carathèodory set of cardinality i, with $\{v_{n-5}, v_{n-4}\} = \delta H(S_i)$, where $n = 3i$.

Take $S_{i+1} = S_i \cup \{v_n\}$, and note that all vertices in S_{i+1} are connected to v_4, and that any other vertex in $V(G_{i+1})$ is connected to at least one vertex in S_{i+1}, by construction of G_{i+1}. Therefore, the longest geodesic of G_{i+1} is at most 4. There are three edges connecting the vertices in $V(G_i)$ to those new vertices added to build G_{i+1}, namely $v_{n-4}v_{n-2}, v_{n-3}v_{n-2}$, and v_4v_n. Choose any two vertices x, y in $V(G_i)$, other than v_4 (note that the geodesic distance of any vertex to v_4 is at most 2). If you take a path between x and y that uses both edges $v_{n-4}v_{n-2}$ and $v_{n-3}v_{n-2}$, it is bigger than their geodesic in at least one unit, since $v_{n-4}v_{n-3} \in E(G_i)$. If you take a path between x and y that uses $v_{n-4}v_{n-2}$ or $v_{n-3}v_{n-2}$, and v_4v_n, this path is bigger than 4. So, by inductive hypothesis, in G_{i+1} no shorten path was introduced between two vertices of G_i, implying that $H(S_i)$ in G_{i+1} is exactly $V(G_i)$.

All the geodesics between v_n and another vertex in S_{i+1} have size 2, and between v_n and another vertex not in S_{i+1} have size at most 3. Take any vertex in $V(G_i)$ that is not in S_{i+1}, other than v_{n-4}. If a path between such vertex and v_n uses either edge $v_{n-4}v_{n-2}$ or $v_{n-3}v_{n-2}$, it is bigger than 3, so it is not a geodesic. The geodesics between v_{n-4} and v_n are $v_{n-4}v_{n-2}v_{n-1}v_n$ and $v_{n-4}v_{n-3}v_4v_n$. This means the only minimum path between a vertex in $V(G_i)$ and v_n that uses the vertices v_{n-2} and v_{n-1} is the one between v_{n-4} and v_n. Without loss of generality, using the inductive hypothesis, we can conclude that $\delta H(S_{i+1}) = \{v_{n-2}, v_{n-1}\}$ (meaning S_{i+1} is a Carathèodory set), and $|S_{i+1}| = i + 1$, completing the proof. □

Corollary 3.2. *[5] The Carathèodory number on geodesic convexity is unlimited.*

Proof. By Thereom 3.1, for all integer $i \geq 1$, there is a Carathèodory set of cardinality i in G_i. We conclude that $C(G_i) \geq i$ and thus yields the results. □

Theorem 3.3. *[3] Carathèodory Number is NP-complete in geodetic convexity*

Proof. Using shortest paths algorithms, the convex hull of a set of vertices of some graph can be determined efficiently. Therefore, it can be checked in polynomial time whether a given set of vertices is a Carathèodory set, which implies that Carathèodory Number in geodetic convexity is in NP.

□

Theorem 3.4. *[3] Local Caratheodory Number is NP-complete in geodetic convexity even restricted to bipartite graphs*

Proof. Similarly as in the proof of Theorem 3.3, it follows easily that Local Caratheodory Number in geodetic convexity is in NP. In order to complete the proof, we describe a polynomial reduction of the problem Geodetic Hull Number restricted to instances (G, k) where G is a bipartite graph with a vertex of degree 1, to Local Caratheodory Number. Note that Geodetic Hull Number restricted to such instances is still NP-complete

Geodetic Hull Number Instance: A graph G and an integer k. Question: Is there a set S of at most k vertices of G with $H(S) = V(G)$ in geodetic convexity? Let (G, k) be such a restricted instance of Geodetic Hull Number. Let $V(G) = v_1, \cdots, v_n$ and let $d_G(v_1) = 1$. We describe the construction of an instance (G', U', u', k') of Local Caratheodory Number.

- Add to G n new vertices $w_1, \cdots, w_n$.
- Add all edges w_iv_i for $i \in \{1, ..., n\}$ and w_iw_{i+1} for $i \in \{1, ..., n-1\}$.
- Subdivide each edge w_iw_{i+1} exactly $dist_G(v_i, v_{i+1}) - 1$ times.

This completes the description of G'. Note that G' is bipartite.

Let $U' = (V(G) \setminus \{v_1\}) \cup \{w_1\}$, $u' = w_n$, and $k' = k$.

Since we are considering convex hulls in the geodetic convexities induced by the two graphs G and G', we will add an index indicating the corresponding graph. It remains to prove that there is a set S of at most k vertices of G with $H_G(S) = V(G)$ if and only if there is a subset F of U' of order at most k with $w_n \in H_{G'}(F)$.

We first prove the necessity. Let S be a set of at most k vertices of G with $H_G(S) = V(G)$. Since v_1 has degree 1, we have $v_1 \in S$. If $F = (S \setminus \{v_1\}) \cup \{w_1\}$, then $|F| = |S| \leq k$. Furthermore, for every shortest path $x_0 \cdots x_l$ in G between $v_1 = x_0$

and $v_i = x_l$, the path $w_1 x_0 \cdots x_l$ is a shortest path in G' between w_1 and $v_i = x_l$. This implies that $V(G) \subseteq H_{G'}(F)$ and, by construction, $w_n \in V(G') = H_{G'}(F)$. Now we proceed to the proof of the sufficiency. Let F be a subset of U' of order at most k with $w_n \in H(F)$. Note that no vertex in $\{w_1, \cdots, w_n\}$ lies on a shortest path in G' between two vertices in $V(G)$. Since $w_n \in H(F)$, this implies $w_1 \in F$. Since the geodetic convexity is an interval convexity, we may consider its interval function I as in

$$I(\{x,y\}) = \{x,y\} \cup \{z \mid z \ lies \ on \ a \ path \ in \ P \ between \ x \ and \ y\} \tag{3.1}$$

Furthermore, we consider its iterated interval function I^k as in

$$I^k(U) := I(I^{k-1}(U)) \ for \ every \ k \geq 2. \tag{3.2}$$

. For $i \in \{1, \cdots, n\}$, let $k_i = min\{k \mid w_i \in I^k(F)\}$ where $min\ \emptyset = \infty$. Since $w_1 \in F$ and $w_n \epsilon H(F)$, we have $k_1 = 0$ and $k_n < \infty$. Since w_n has degree 2 and w_i has degree 3 for $i \in \{2, \cdots, n-1\}$, it follows easily by an inductive argument that $k_1 \leq k_2 \leq \cdots \leq k_n$, that is, all k_i are finite, which implies $\{w_1, ..., w_n\} \subseteq H_{G'}(F)$. By construction, if some vertex w_i lies on a shortest path in G' between some vertex w_s and some vertex v_t, then v_i also lies on a shortest path in G' between w_s and v_t. This implies $\{v_1, ..., v_n\} \subseteq H_{G'}(F)$, that is $V(G') = H_{G'}(F)$. Furthermore, it implies that $S = (F \setminus \{w_1\}) \cup v1$ is a set of at most k vertices of G with $V(G) = H_G(S)$, which completes the proof. □

Definition 3.1.3. A graph G is a **split graph** if its vertex set can be partitioned into a complete set and an independent set.

Theorem 3.5. *[3] The Carathèodory number of the geodetic convexity of a split graph is at most 3*

Proof. Let G be a split graph. Let $V(G) = C \cup I$ where C is a complete set and I is an independent set with $C \cap I = \emptyset$. Let U be a Carathèodory set of G of maximum cardinality. We will prove that U contains at most 3 elements. Let $v \in \delta H(U)$.

As before we consider the interval function I as in (3.1) and the iterated interval function I^k as in (3.2). Let k be minimal with $v \in I^k(U)$. If $k \leq 1$, then either v belongs to U or v lies on a shortest path between two vertices in U, which implies $|U| \leq 2$. Hence we may assume that $k \geq 2$.

Since no vertex in I lies on a shortest path between two other vertices, the set $H(U) \setminus U$ contains no element of I. This implies $v \in C$. Since $k \geq 2$, the vertex v lies on a shortest path between two vertices u_{k-1} and w_k in $I^{k-1}(U)$. Necessarily, one of the two vertices, say w_k, belongs to $U \cap I$. Furthermore, by the minimality of k, the other vertex u^{k-1} belongs to $I^{k-1}(U) \setminus I^{k-2}(U)$ and hence $u_{k-1} \in C$. Similarly, since $k \geq 2$, the vertex u_{k-1} lies on a shortest path between two vertices u_{k-2} and w_{k-1} in $I^{k-2}(U)$. As before, we may assume that $w_{k-1} \in U \cap I$. If $k \geq 3$, then this implies that u_{k-2} belongs to $I^{k-2}(U) \setminus I^{k-3}(U)$. Iterating this argument, we obtain vertices $u_0, ..., u_{k-1}$ and $w_1, ..., w_k$ with

- $u_0 \in U, u_1, ..., u_{k-1} \in C$, $w_1, ..., w_k \in U \cap I$, and
- $u_i \in I(\{u_{i-1}, w_i\})$ and $u_i \in I^i(U) \setminus I^{i-1}(U)$ for $i \in \{1, ..., k-1\}$.

Since $v \in H(\{u_0, w_1, ..., w_k\})$, we may assume that $k \geq 3$. By the minimality of k, the vertex u_i is a neighbor of w_k for $i \in \{1, ..., k-2\}$, since otherwise $v \in I(\{u_i, w_k\})$ and hence $v \in I^{i+1}(U)$, which is a contradiction. Similarly, the vertex u_i is a neighbor of w_{k-1} for $i \in \{1, ..., k-3\}$, since otherwise $u_{k-1} \in I(\{u_i, w_{k-1}\})$ and hence $v \in I^{i+2}(U)$, which is a contradiction.

If $k = 3$, then $u_1 \in I(\{w_1, w_3\})$, $u_2 \in I(\{u_1, w_2\})$, and $v \in I(\{u_2, w_3\})$, which implies $v \in H(\{w_1, w_2, w_3\})$ and hence $|U| = 3$.

If $k \geq 4$, then $u_{k-3} \in I(\{w_{k-1}, w_k\})$, $u_{k-2} \in I(\{u_{k-3}, w_{k-2}\})$, $u_{k-1} \in I(\{u_{k-2}, w_{k-1}\})$, and $v \in I(\{u_{k-1}, w_k\})$, which implies $v \in H(\{w_{k-2}, w_{k-1}, w_k\})$ and hence $|U| = 3$, which completes the proof. $\square$

3.2 Helly Number and Radon Number

Definition 3.2.1. A subset $A \subset V$ of a graph $G = (V, E)$ is called **Helly independent** if $\bigcap_{a \in A}[A - a] = \emptyset$.

Definition 3.2.2. A subset $A \subset V$ of a graph $G = (V, E)$, a partition $\{A_1, A_2\}$ of A is called a **Radon partition** if $[A_1] \cap [A_2] \neq \emptyset$.

Definition 3.2.3. A subset $A \subset V$ of a graph $G = (V, E)$ is called **Radon dependent** if it has a Radon partition, and otherwise it is said to be **Radon independent**.

Definition 3.2.4. The **Helly number** $H(G)$ of a graph G is the maximum cardinality of a Helly independent set. H(G) is the smallest integer such that every family of convex sets with an empty intersection contains a subfamily of atmost $H(G)$ members with an empty intersection.

Definition 3.2.5. The **Radon number** $R(G)$ of a graph $G = (V, E)$ is the maximum cardinality of a Radon independent set.

Theorem 3.6. *[6] Let $G = (V, E)$ be a connected graph of order $n \geq 2$.*

1. *Every subset of a Helly independent set is Helly independent.*

2. *Every subset of a Radon independent set is Radon independent.*

3. *$2 \leq \omega(G) \leq H(G)$.*

4. *$H(G) \leq R(G)$.*

Proof. 1. Take $B \subset A \subseteq V$. Suppose that B is Helly dependent. Observe that

$$\cap_{b \in B}[B - b] \subseteq \cap_{b \in B}[A - b]$$

Note also that, for every $a \in A - B$,

$$\cap_{b \in B}[B - b] \subseteq [B] \subseteq [A - a]$$

Hence,

$$\emptyset = \cap_{b\in B}[B-b] \subseteq (\cap_{b\in B}[A-b]) \cap (\cap_{a\in A} - B[A-a]) = \cap_{a\in A}[A-a]$$

2. Take $B \subset A \subseteq V$. Suppose that B is Radon dependent. Let $\{B_1, B_2\}$ be a Radon partition of B. Consider the partition of A, $\{B_1, A_2\}$, where $A_2 = B_2 \cup (A - B)$. Then, as

$$\emptyset = [B_1] \cap [B_2] \subseteq [B_1] \cap [A_2]$$

it follows that $\{B_1, A_2\}$ is a Radon partition of A.
3. Let $A \subseteq V$ such that $G[A] \cong Kh$. Then, for every $a \in A$, $G[A-a] \cong K_h - 1$, which means that $\cap_{a\in A}[A-a] = \emptyset$.
4. Let $A \subseteq V$ be Radon dependent. Consider a Radon partition $\{A_1, A_2\}$ of A and a vertex $p \in [A_1] \cap [A_2]$. For each $a \in A$, we have either $A_1 \subseteq A - a$ or $A_2 \subseteq A - a$. Hence, $p \in [A-a]$ for every $a \in A$, which means that A is Helly dependent. □

Lemma 3.7. *[2] Let G be a graph. If $S \subseteq V(G)$ is a Helly (or a Radon) independent set with at least 3 vertices, then the subgraph of G induced by S is a union of complete graphs. Furthermore, if G is bipartite, then S is an independent set.*

Proof. Let $S \subseteq V(G)$ be any set such that the subgraph of G induced by S is not a union of complete graphs. Then, there exist vertices $u, v, w \in S$ such that $u, v \notin E(G)$ and

$$w \in N(u) \cap N(v)$$

Hence

$$w \in I(u, v)$$

This implies that $(\{w\}, S\backslash\{w\})$ is a Radon partition of S. Then, S is not a Radon independent set.

Now, consider that G is bipartite with bipartition (A, B) and let $S \subseteq V(G)$ containing adjacent vertices a and b. Without loss of generality, we can say that $a \in A$ and there exists $a' \in (A \cap S) \setminus \{a\}$. It is clear that

$$d(a', b) - 1 \leq d(a', a) \leq d(a', b) + 1$$

This implies that $a \in I(a', b)$ or $b \in I(a', a)$. As we have seen above, this means that S is not a Radon independent set.

To complete the proof it suffices to observe that every Helly independent set S is also Radon independent, because $[S_1] \cap [S_2] \subseteq \bigcap_{v \in S}[S \setminus \{v\}]$ for any partition (S_1, S_2) of S □

Lemma 3.8. *[2] Let $u, v,$ and w be vertices of a bipartite graph G. If $uv \in E(G)$, then either $u \in I(v, w)$ or $v \in I(u, w)$.*

Proof. First, we claim that $d(u, w) \neq d(v, w)$.

Thus, suppose for contradiction that $d(u, w) = d(v, w)$. Let P_{uw} and P_{vw} be geodesics beginning at u and v, respectively, and finishing at w. Denote by w' the most distant vertex of w that belongs to both P_{uw} and P_{vw}. Now, let $P_{uw'}$ and $P_{vw'}$ be the subpaths of P_{uw} and P_{vw} beginning at u and v, respectively, and finishing at w'. It is clear that $P_{uw'}$ and $P_{vw'}$ have the same length and, therefore, the concatenation of $P_{uw'}$, $P_{vw'}$, and uv form an odd cycle.

This is a contradiction because G is a bipartite graph. Hence, $d(u, w) \neq d(v, w)$ as claimed. Now, it remains to notice that if $d(u, w) < d(v, w)$, then $u \in I(v, w)$ and $v \notin I(u, w)$. □

Lemma 3.9. *[2] If u and v are vertices of a Helly independent set S of a bipartite graph, then every (u, v)-geodesic contains a vertex w such that $w \in I(x, y)$ for $x, y \in S$ only if $\{x, y\} = \{u, v\}$*

Proof. Let S be a Helly independent set of a bipartite graph G. The result is

trivial for $|S| \leq 2$. Then, we can assume $|S| \geq 3$. Suppose for contradiction that $P_{uv} = ua_1...a_tv$ is a (u,v)-geodesic for $u, v \in S$ such that, for every $i \in \{1, ..., t\}$, $a_i \in I(x_i, y_i)$ for some $\{x_i, y_i\} \subset S$ and $\{x_i, y_i\} \neq \{u, v\}$. By Lemma 3.7, S is an independent set, which means that $t \geq 1$. Observe that if $\{x_i, y_i\} \cap \{u, v\} = \emptyset$ for some $i \in \{1, ..., t\}$, then $a_i \in [S \setminus \{w\}]$ for every $w \in S$, which contradicts the assumption that S is Helly independent. Therefore, for every $i \in \{1, ..., t\}$, we assume $x_i \in \{u, v\}$.First, consider $x_t = u$. Then, $a_t \in I(u, y_t)$.
If $a_t \in I(v, y_t)$ then, $a_t \in [S \setminus \{w\}]$, for every $w \in S$, which is not possible, because S is Helly independent. Therefore, $a_t \notin I(v, y_t)$ and, since G is bipartite, Lemma 3.8 guarantees that $v \in I(a_t, y_t)$. But this means that $v \in [S \setminus \{w\}]$, for every $w \in S$, which contradicts the assumption that S is a Helly independent set. Then, we can assume $x_t = v$. By symmetry, we can assume $x_1 = u$. Let k be the minimum positive integer such that $x_k = v$. It is clear that $1 < k \leq t$. Furthermore, by Lemma 3.8 again, either $a_{k-1} \in I(a_k, y_k)$ or $a_k \in I(a_{k-1}, y_k)$. In the former case, we have $a_{k-1} \in [S \setminus \{w\}]$ for every $w \in S$, while in the latter case we have $a_k \in [S \setminus \{w\}]$ for every $w \in S$, which is a contradiction. □

Theorem 3.10. *[2] If G is a bipartite graph or order $n \geq 3$, then $H(G) \leq \frac{-1+\sqrt{1+8n}}{2}$.*

Proof. Suppose for contradiction that $H(G) = k > \frac{-1+\sqrt{1+8n}}{2} \geq 2$. Then, G contains a Helly independent set S with k vertices. Write $S = \{v_1, ..., v_k\}$. By Lemma 3.9, for each pair $u, v \in S$ there exists a vertex $w \in I(u, v)$ which does not appear in the interval between any other pair of vertices of S. Which means that the number of vertices of G is at least the size of S plus the number of pairs of S, i.e.,

$$n \geq k + \frac{k^2 - k}{2} = \frac{k^2 + k}{2}$$

Since

$$k > \frac{-1 + \sqrt{1 + 8n}}{2} \geq 2$$

we have

$$
\begin{aligned}
n &> \frac{\left(\frac{-1+\sqrt{1+8n}}{2}\right)^2 + \frac{-1+\sqrt{1+8n}}{2}}{2} \\
&= \frac{\frac{-2\sqrt{1+8n}+2+8n}{4} + \frac{-1\sqrt{1+8n}}{2}}{2} \\
&= \frac{\frac{2+8n-2}{4}}{2} \\
&= n
\end{aligned}
$$

which is a contradiction, completing the proof.

□

Theorem 3.11. *[2] A bipartite graph G of order $n \geq 3$ has Helly number $\frac{-1+\sqrt{1+8n}}{2}$ if and only if $G \in \mathcal{H} = \{G_t : t \geq 2\}$.*

Proof. First, we show that every $G_t \in \mathcal{H}$ for $t \geq 2$ satisfies $H(G_t) = t$ where $t = \frac{-1+\sqrt{1+8n}}{2}$ and $n = |V(G_t)|$. By Theorem 3.10, it suffices to show thatG_t contains a Helly independent set with t vertices. We show that $A_t = \{v_1, ..., v_t\}$ is a Helly independent set. Denote $S_k = A_t \setminus \{v_k\}$ for $1 \leq k \leq t$. By the construction of G_t, it is easy to see that, for every k, it holds $[S_k] = S_k \cup \{v_{ij} : v_i, v_j \in S_k\}$. This implies that, for every k, $v_k \notin [S_k]$, and for every pair $i, j \in \{1, ..., t\}$, $v_{ij} \notin [S_i] \cup [S_j]$.Therefore,

$$\bigcap_{k=1}^{t} [S_k] = \emptyset$$

and A_t is indeed a Helly independent set.

Conversely, let G be a bipartite graph of order $n \geq 3$ with bipartition (A, B) whose Helly number is $t = \frac{-1+\sqrt{1+8n}}{2}$. Then, G contains a Helly independent set S of size t. By Lemma 3.7, S is an independent set. Furthermore, by Lemma 3.9, for every pair of vertices $a, b \in S$ there exists a vertex v_{ab} that belongs to $I(a, b)$ and does not belong to the interval of any other pair of vertices of S. Since $n = \frac{t^2+t}{2}$, the number of vertices of $V(G) \setminus S$ is equal to the number of pairs of S. From these facts, we conclude that $I(a, b) = \{a, b, v_{ab}\}$ and $N(v_{ab}) = \{a, b\}$. Furthermore, S is contained in one part of G, say A, and all other vertices of G are in B. Now, it is clear that G is isomorphic to G_t. □

Lemma 3.12. *[2] If u, u' are vertices of a Radon independent set S of a bipartite graph, then every (u, u')-geodesic contains a vertex w such that $w \in I(u'', x) \cap I(u'', y)$ for $u'' \in \{u, u'\}$ and $x, y \in S \setminus \{u''\}$ only if $\{u'', x\} = \{u, u'\}$ or $\{u'', y\} = \{u, u'\}$.*

Proof. Let S be a Radon independent set of a bipartite graph G. The result is trivial for $|S| \leq 3$. Then, we can assume $|S| \geq 4$. Suppose for contradiction that $P_{uu'} = ua_1...a_tu'$ is a (u, u')-geodesic for $u, u' \in S$ such that, for every $i \in \{1, ..., t\}$, there exist $u''_i \in \{u, u'\}$ and $x_i, y_i \in S \setminus \{u, u'\}$ satisfying $a_i \in I(u''_i, x_i) \cap I(u''_i, y_i)$. By Lemma 3.7, $t \geq 1$.

First, assume $u''_t = u$. Then, u', x_t, y_t are pairwise distinct.

If $a_t \in I(u', x_t)$, then $a_t \in [u, y_t] \cap [u', x_t]$. Then, $a_t \notin I(u', x_t)$ and, since G is bipartite, by Lemma 3.8, $u' \in I(a_t, x_t)$. This means that $u' \in [S \setminus \{u'\}]$, which is not possible because S is a Radon independent set. Therefore, $u''_t = u'$. By a symmetric argument, we conclude that $u''_1 = u$.

Let k be the minimum positive integer such that $u''_k = u'$. It is clear that $1 < k \leq t$. Furthermore, by Lemma 3.8 again, either $a_{k-1} \in I(a_k, y_k)$ or $a_k \in I(a_{k-1}, y_k)$. In the former case, we would have $a_{k-1} \in [x, u] \cap [y_k, u']$ for $x \in \{x_{k-1}, y_{k-1}\} \setminus \{y_k\}$. Therefore, $a_k \in I(a_{k-1}, y_k)$.

If $x_k \neq x_{k-1}$, then $a_k \in [u, x_{k-1}, y_k] \cap [x_k, u']$ even if $x_{k-1} = y_k$. Which means $x_k = x_{k-1}$. If $a_k \in I(a_{k-1}, x_{k-1})$, then $a_k \in [u, x_{k-1}] \cap [u', y_k]$. Hence, $a_{k-1} \in I(a_k, x_k)$. However, this implies that $a_{k-1} \in [y_{k-1}, u] \cap [x_k, u']$, which is a contradiction. $\square$

Corollary 3.13. *[2] Let S be a Radon independent set of a bipartite graph with $|S| \geq 3$. For every $u, u' \in S$, there is a vertex in $I(u, u') \setminus \{u, u'\}$ appearing in at most three intervals among the pairs of S.*

Proof. For $|S| = 3$, it is trivial. Then, assume $|S| \geq 4$. First, we recall that if $w \in I(u, u') \setminus \{u, u'\}$, then the fact that S is Radon independent set guarantees that every pair of vertices of S, whose interval contains w, intersects u, u'.

Now, since u belongs to at least three pairs of S, by Lemma 3.12, there is a vertex $w \in I(u, u') \setminus \{u, u'\}$ such that if there exists $x \in S \setminus \{u, u'\}$ with $w \in I(u'', u') \cap I(u'', x)$ for $u'' \in \{u, u'\}$, then there is no $y \in S \setminus \{u, u', x\}$ such that $w \in I(u'', y)$. Therefore, if no such x exists for $w \in I(u, u') \setminus \{u, u'\}$, then the only interval among the pairs of S containing w is $\{u, u'\}$ and we are done. Otherwise, without loss of generality, we can assume that $\{u, u'\}$ and $\{u, x\}$ are the only pairs among the vertices of S containing u whose intervals contain w.

Next, suppose that there is $y' \in S \setminus \{u, u'\}$ such that $w \in I(u', u) \cap I(u', y')$. If $y' \neq x$, then $w \in [u, x] \cap [u', y']$, which contradicts the assumption that S is a Radon independent set. Then, $y' = x$ and the only pairs among the vertices of S containing u', whose intervals contain w, are $\{u, u'\}$ and $\{u', x\}$. Hence, w appears in at most three intervals among the pairs of S. $\square$

Theorem 3.14. *[2] If G is a bipartite graph or order $n \geq 3$, then $R(G) \leq \frac{-5+\sqrt{25+24n}}{2}$.*

Proof. Suppose for contradiction that G is a bipartite graph with $r(G) = k > \frac{-5+\sqrt{25+24n}}{2}$. Then, G contains a Radon independent set S with k vertices. Write $S = \{v_1, ..., v_k\}$. By Lemma 3.7, S is an independent set. Then, each of the $\frac{k^2-k}{2}$ intervals of the pairs of S contains a vertex not in S. Furthermore, Lemma 3.12 guarantees that, for each such a pair, there is a vertex in $V(G) \setminus S$ that belongs to at most 3 of such intervals. Therefore, the number of vertices of G is at least the size of S plus the number of pairs of S divided by three, i.e.,

$$n \geq k + \frac{\frac{k^2-k}{2}}{3} = \frac{k^2+5k}{6}$$

Now since

$$k > \frac{-5+\sqrt{25+24n}}{2} \geq 2$$

we have

$$\begin{aligned}
n &\geq \frac{k^2+5k}{6} \\
&> \frac{\left(\frac{-5+\sqrt{25+24n}}{2}\right)^2 + 5\left(\frac{-5+\sqrt{25+24n}}{2}\right)}{6} \\
&= \frac{\frac{25-10\sqrt{25+24n}+25+24n}{4} - \frac{25}{2} + \frac{5\sqrt{25+24n}}{2}}{6} \\
&= \frac{\frac{24n}{4}}{6} \\
&= n
\end{aligned}$$

which is a contradiction, completing the proof. $\square$

Conclusion

This dissertation has examined geodesic convexity in graphs and several parameters that arise naturally within this framework. The study began with the essential preliminaries of graph theory and graph convexity, including shortest paths, geodesic intervals, and geodesically convex sets. These notions provided the conceptual foundation for the remainder of the dissertation and established the setting in which the later results were discussed. By introducing these basic ideas at the outset, the dissertation situated geodesic convexity within the broader context of discrete metric structure rather than treating it as an isolated notion.

A principal focus of the dissertation was the geodesic number, hull number, and convexity number. These invariants describe different aspects of the structure of a graph under geodesic convexity. The geodesic number measures the minimum size of a set that generates the graph through geodesic closure, the hull number determines the smallest set whose convex hull is the entire graph, and the convexity number identifies the size of a largest proper convex subset of the graph. The dissertation further considered the Caratheòdory, Helly, and Radon numbers, which are classical parameters in convexity theory and have natural analogues in graph convexity.These are classical parameters from convexity theory that have also been extended to graph convexities.

One of the main conclusions of this work is that geodesic convexity provides a strong link between metric and structural graph theory. Because the convexity is defined through shortest paths, its behavior is highly sensitive to the geometry of the graph. This gives the subject a distinctly combinatorial as well as geometric character. The parameters considered in this dissertation capture different facets of that structure, and their interaction shows that even graphs with simple definitions may exhibit subtle and varied convexity behavior. In this respect, geodesic convexity serves as a useful framework for understanding both local and global structural features of graphs.

The study of geodesic convexity in graphs offers substantial opportunities for further research. A natural extension of the present dissertation is the examination of the parameters studied here in additional graph classes. While this dissertation has introduced the general theory and discussed the principal invariants, many families of graphs remain to be analyzed in detail. In particular, bridged graphs, chordal graphs, perfect graphs, meshed graphs, and graph products appear to be promising settings for future investigation, since their structural properties may lead to sharper results or exact parameter values.

Another direction concerns the computational complexity of the invariants studied in this dissertation. Determining the geodesic number, hull number, or convexity number is often nontrivial, and related graph problems are known to be computationally difficult in general. Future work may therefore focus on the design of efficient algorithms for restricted graph classes, the proof of hardness results for broader classes, or the development of approximation methods. Such research would strengthen the algorithmic dimension of graph convexity and increase its relevance in computational graph theory.

In addition, computational experimentation offers another useful direction for future work. Software-based exploration of small and medium-sized graphs can assist in computing convex hulls, identifying extremal sets, and testing conjectures about the parameters studied in this dissertation. Since several graph convexity problems are computationally complex, such experiments can complement theoretical analysis and help identify patterns that motivate further proof.A further direction is to examine how geodesic convexity interacts with other convexities, such as monophonic or P_3 convexity, in order to obtain a broader picture of discrete convexity in graphs. These possibilities show that the topic has both theoretical depth and scope for future work. Overall, this dissertation provides a foundation for further study of convexity-related parameters in graphs and their applications in graph theory.

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